PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES, INDIA

1966

VOL. XXXVI SECTION—A PART I

NATIONAL ACADEMY OF SCIENCES, INDIA
ALLAHABAD

The *Proceedings of the National Academy of Sciences, India,* is published in two sections : Section A (Physical Sciences) and Section B (Biological Sciences). Four parts of each section are published annually (since 1960).

The Editorial Board in its work of examining papers received for publication is assisted, in an honorary capacity, by a large number of distinguished scientists. Papers are accepted from members of the Academy *in good standing*. In case of a joint paper, one of the authors must be a member of the Academy. The Academy assumes no responsibility for the statements and opinions advanced by the authors. The papers must conform strictly to the rules for publication of papers in the *Proceedings*. A total of 50 reprints is supplied free of cost to the author or authors. The authors may have any reasonable number of additional reprints at cost price, provided they give prior intimation while returning the proof.

Communications regarding contributions for publication in the *Proceedings*, books for review, subscriptions etc. should be sent to the General Secretary, National Academy of Sciences, India, Lajpatrai Road, Allahabad-2 (India).

Annual Subscription for both Sections : Rs. 65 ; for each Section : Rs. 35 (including Annual Number) ; Single Copy: Rs. 7.50 ; Annual Number Rs. 5

PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
INDIA
1966

VOL. XXXVI SECTION—A PART I

ON APPLICATION OF THE OPERATIONAL METHODS TO THE BESSEL COEFFICIENTS OF TWO ARGUMENTS

By
S. L. GUPTA
Lecturer in Mathematics, University of Jodhpur, Jodhpur

[Received on 1st February, 1964]

ABSTRACT

In this note operational images of the Bessel coefficients of two arguments have been given and from the operational methods two results have been obtained.

1. Operational images.

Taking $J_n(x, y)$ the Bessel coefficient of two arguments x, y and of integral order n defined by the equality

$$e^{\frac{1}{2}x\left(u-\frac{1}{u}\right)+\frac{1}{2}y\left(u^2-\frac{1}{u^2}\right)}=\sum_{n=-\infty}^{\infty} u^n J_n(x, y), \tag{1}$$

we find that

$$J_n(x, y) = \sum_{k=-\infty}^{\infty} J_{n-2k}(x) J_k(y). \tag{2}$$

So that

$$J_{2n}(x, y) = \sum_{k=o}^{n} J_{2n-2k}(x) J_k(y) + \sum_{k=n+1}^{\infty} J_{2k-2n}(x) J_k(y) + \sum_{k=1}^{\infty} (-1)^k J_{2n+2k}(x) J_k(y),$$

$$J_{2n+1}(x, y) = \sum_{k=o}^{n} J_{2n+1-2k}(x) J_k(y) - \sum_{k=n+1}^{\infty} J_{2k-2n-1}(x) J_k(y) + \sum_{k=1}^{\infty} (-1)^k J_{2n+2k+1}(x) J_k(y).$$

On following the notifications of Mclachlan and Humbert 4, p. 62 *i.e.* if

$$\phi(p, q) = pq \int_0^{\infty}\int_0^{\infty} e^{-px-qy} f(x, y)\, dx\, dy$$

then

$$f(x, y) \supset\supset \phi(p, q),$$

where p, q are positive and the integral on the right converges ; we find that

$$J_{2n}(x, y) \supset\supset \frac{p}{r}\frac{q}{s}\left\{\frac{P^{2(n+1)} - Q^{(n+1)}}{P^{2n}\, Q^n\,(P^2 - Q)} + \frac{P^2 Q\,(P^{2n} - Q^n) + (P^{2n} + Q^n)}{P^{2n}\, Q^n\,(P^2 Q - 1)\,(P^2 Q + 1)}\right\}, \tag{3}$$

$$J_{2n+1}(x, y) \supset\supset \frac{p}{r}\frac{q}{s}\left\{\frac{1}{P^{2n+1} Q^n}\left(\frac{P^{2(n+1)} - Q^{(n+1)}}{P^2 - Q}\right) - \frac{P^2\, Q\,(P^{2(n+1)} + Q^n) + (P^{2(n+1)} - Q^n)}{P^{2n+1}\, Q^n\,(P^2\, Q - 1)\,(P^2\, Q + 1)}\right\}$$

where

$$P = p + \sqrt{p^2+1},\ Q = q + \sqrt{q^2+1},\ r = \sqrt{p^2+1} \text{ and } s = \sqrt{q^2+1}$$

Form (3) and (4) the symmetry in the operational images is evident [3, p. 20].

2. Evaluation of a definite integral involving $J_0(2\sqrt{ax}, x)$.

Form (2) we have

$$J_0\,(2\sqrt{ax}, x) = J_0(x)\, J_0(2\sqrt{ax}) + 2\sum_{k=1}^{\infty} J_{2k}(x)\, J_{4k}(2\sqrt{ax}),$$

taking a to be positive, for $R(p) > 0$, [1, p. 186 (28)], we have

$$J_{2k}(x) J_{4k}(2\sqrt{ax}) \supset \frac{p}{\sqrt{p^2+1}}\, e^{\frac{-ap}{p^2+1}}\, J_{2k}\left(\frac{a}{p^2+1}\right).$$

Therefore

$$J_0(2\sqrt{ax}, x) \supset \frac{p}{\sqrt{p^2+1}}\, e^{\frac{-ap}{p^2+1}}\left(J_0\left(\frac{a}{p^2+1}\right) + 2\sum_{k=1}^{\infty} J_{2k}\left(\frac{a}{p^2+1}\right)\right)$$

$$= \frac{p}{\sqrt{p^2+1}}\, e^{\frac{-ap}{p^2+1}}. \tag{5}$$

Now, let us take into account the result [1, p. 138]

$$\frac{1}{\sqrt{x^2+1}} \supset \tfrac{1}{2}\pi p\,(\mathbf{H}_0(p) - Y_0(p)),$$

so that, by the application of Goldstein's theorem [2, p. 103],
we get

$$\tfrac{1}{2}\pi\int_0^{\infty} J_0\,(2\sqrt{ax}, x)\,(\mathbf{H}_0(x) - Y_0(x))\, dx = \int_0^{\infty}\frac{e^{\frac{-ax}{x^2+1}}}{x^2+1}\, dx = \int_0^{\pi/2} e^{-\frac{1}{2}a\sin\theta}\, d\theta$$

$$= \tfrac{1}{2}\pi\,(I_0(\tfrac{1}{2}a) - \mathbf{L}_0(\tfrac{1}{2}a)),\ [6, \text{p. } 338].$$

Hence, we find that

$$\int_0^\infty J_0(2\sqrt{ax}, x)\,(\mathbf{H}_0(x) - Y_0(x))\,dx = (I_0(\tfrac{1}{2}a) - \mathbf{L}_0(\tfrac{1}{2}a)). \tag{6}$$

3. A self reciprocal property of $x^{\frac{1}{2}} J_0\ (x\sqrt{2a},\ \frac{1}{2}x^2)$

If we adopt the usual notation of Hardy and Titchmarsh [5, p. 245], we say a function $f(x)$ to be R_ν if it is self-reciprocal in Hankel transform of order ν, *i.e.*, if

$$f(x) = \int_0^\infty (x\,y)^{\frac{1}{2}}\, J_\nu\,(x\,y)\, f(y)\ dy.$$

Also, when

$f(x) \supset \phi(p)$, then we have [4, p. 12]

$$p^{1-\nu}\ \phi\left(\frac{1}{p}\right) \subset x^{\frac{1}{2}\nu} \int_0^\infty t^{-\frac{1}{2}\nu}\, J_\nu\ (2\sqrt{xt})\ f(t)\,dt.$$

And if

$$\phi(p) = p^{1-\nu}\,\phi\left(\frac{1}{p}\right) \text{ then } x^{-\nu+\frac{1}{2}} f(\tfrac{1}{2}x^2) \text{ is } R_\nu.$$

Let us take

$f(x) = J_0(2\sqrt{ax},\ x)$,

so that from (5), we have $\phi(p) = \dfrac{p}{\sqrt{p^2+1}}\, e^{\frac{-ap}{p^2+1}}$

Here, $p^{1-\nu}\ \phi\left(\dfrac{1}{p}\right) = \dfrac{p}{\sqrt{p^2+1}}\, e^{\frac{-ap}{p^2+1}} = p^{-\nu}\,\phi(p)$, therefore, it implies that

$$x^{1/2}\, J_0\,(x\,\sqrt{2a},\ \tfrac{1}{2}x^2) \text{ is } R_0.$$

ACKNOWLEDGMENT

The author is thankful to Dr. S. C. Mitra for his kind interest during the preparation of this note.

REFERENCES

1. Erdelyi, A. *et al.* Tables of integral transforms, Vol. 1, §§ 5·2, 5·4, (1954).
2. Goldstein, S. *Proc. London Math. Soc.*, (2) 34, 103, (1932).
3. Humbert, P. *Ann. Soc. Sci.* Bruxelles LXVII, 19–22, (1953).
4. Mclachlan and Humbert. *Memorial des Math. Sciences*, 62, 63, (1941).
5. Titchmarsh, E. C. Theory of Fourier Integrals, 2nd Ed., 245, (1948).
6. Watson, G. N. Theory of Bessel Functions, 338, (1944).

PHYSICO-CHEMICAL METHODS FOR ESTIMATION OF ALCOHOLIC AND OTHER CONSTITUENTS IN SYNTHETIC MIXTURES AND NATURAL ESSENTIAL OILS PART XI, TERNARY SYSTEMS CONSISTING OF TWO ALCOHOLS AND ONE ESTER

By

R. N. LAL *and* V. M. PATWARDHAN

Department of Chemical Engineering and Technology
Harcourt Butler Technological Institute, Kanpur

[Received on 8th December, 1964]

ABSTRACT

Experimental verification of the formulae based on specific volume of the synthetic mixture and its individual constituents and ester value of the mixture before acetylation has been studied in the case of ternary mixtures consisting of two alcohols and one ester. The percentage of an individual alcohol could be determined within an accuracy of 3·0 to 8·2 percent and 0·5 to 4·1 percent in the case of two systems depending on the nature of ester present in the mixture.

The earlier workers Kumar *et al.*[1] have theoretically derived formulae 1, 2 and 3 for the determination of the percentage of two alcohols and an ester present in a ternary system.

$$a = \frac{100\,(h_m - h_b) - c\,(h_c - h_b)}{(h_a - h_b)} \qquad \ldots . (1)$$

$$b = \frac{100\,(h_m - h_a) - c\,(h_c - h_a)}{(h_b - h_a)} \qquad \ldots . (2)$$

$$c = \frac{100\,v_1}{V_c} \qquad \ldots . (3)$$

where a, b and c are the respective percentage of alcohols A, B and ester C having molecular weights M_a, M_b and M_c and specific volumes h_a, h_b and h_c respectively. Let V_c be the ester value of pure ester C and h_m and v_1 be the specific volume and ester value respectively of the mixture before acetylation.

In the present investigation, application of these formulae has been studied for the following systems :

(1) Benzyl alcohol, Phenyl ethyl alcohol and Benzyl butyrate system.

(2) Phenyl ethyl alcohol, Phenyl propyl alcohol and Phenyl ethyl acetate system.

EXPERIMENTAL

Preparation of the mixtures.—L. R. grade benzyl alcohol, phenyl propyl alcohol and phenyl ethyl acetate of B. D. H., phenyl ethyl alcohol (Schimmel & Co. Hamburg) and benzyl butyrate (Rhodia) were twice fractionated under reduced pressure using a Tower's fractionating column. The fractions distilling at a constant temperature were collected in amber coloured bottles and their physico-chemical constants are given in Table I. After ensuring cent per cent purity of these chemicals, mixtures having different compositions were prepared.

Determination of specific gravity at t°C.—A double jacketed pyknometer having a thermometer reading upto 0·2°C was used for the determination of specific gravity of the components as well as of the mixtures as previously described[2] by the present worker. The specific gravity was converted into density with the help of standard formula.[3]

Determination of total alcohol after acetylation.—Total alcohols in the mixture have been determined by adding up the percentage of individual alcohols. Further, total alcohols have also been determined by acetylating the mixtures and determining the ester value of the mixture by modified technique.[4]

Acetylation conditions.—Acetylation was carried out with 10 cc. mixture, 15 cc. acetic anhydride and 0·5 gm. fused sodium acetate following the procedure described previously.[5]

In system I consisting of benzyl alcohol, phenyl ethyl alcohol and benzyl butyrate, a slightly high acid value of the acetylated product after thorough washing with sodium chloride containing 2% sodium bicarbonate may be due to liberation of butyric acid from the ester during the acylation process and subsequent incomplete washing.

TABLE I

Physico-Chemical Properties of Chemicals used

Particulars	Benzyl alcohol	Phenyl ethyl alcohol	Phenyl propyl alcohol	Benzyl butyrate	Phenyl ethyl acetate
(1) Sp. gr. 25°/25°C	1·04219	1·02578	—	1·00548	—
Sp. gr, 30°/30°C	—	1·01963	1·00029	—	1·03189
(2) Density 25°/4°C	1·03769	1·01899	—	1·00113	—
Density 30°/4°C	—	1·01665	0·998058	—	1·02888
(3) Sp. Vol. 25°/4°C	0·963680	0·981355	—	0·998870	—
Sp. Vol. 30°/4°C	—	0·983618	1·00195	—	0·971933
(4) Ref. index	1·538 (25°C)	1·531 (20°C)	1·536 (20°C)	1·492 (22°C)	1·498 (20°C)
(5) B. P. °C/mm.	122–123/4	102/10	112–113/7	109/5·5	96/2·5
(6) Acid value	0	0	0	0	0
(7) Ester value, mean					
(*i*) Before acetylation	0	0	0	314·3	341·3
(*ii*) After acetylation	373·5	341·6	314·3	—	—
(8) Oxime number	0	0	0	0	0
(9) % Purity	99·9	99·9	99·8	99·9	99·9

Determination of saponification value by modified technique.—By the modified method,[4] it is possible to determine saponification value with greater accuracy than is possible with the usual method[6] of its determination and was followed,

TABLE II

Benzyl alcohol, Phenyl ethyl alcohol and Benzyl butyrate System.

Mixtures	I	II	III	IV
(1) % Composition				
(*i*) Benzyl alcohol	70·26	49·91	9·69	19·11
(*ii*) Phenyl ethyl alcohol	9·99	30·42	41·22	60·78
(*iii*) Benzyl butyrate	19·75	19·66	49·11	20·11
(2) Sp. gr. 25°/25°C	1·03336	1·02997	1·01727	1·02476
(3) Density 25°/4°C	1·02889	1·02552	1·01289	1·02033
(4) Sp. vol. 25°/4°C	0·971920	0·975116	0·987272	0·980077
(5) Before acetylation				
(*i*) Acid value	0·51	0·43	0·86	0·76
(*ii*) Ester value	63·22	60·83	154·2	63·48
(6) Found %				
(*i*) Benzyl alcohol	73·31	54·45	15·07	27·26
(*ii*) Phenyl ethyl alcohol	6·57	26·23	35·86	52·54
(*iii*) Benzyl butyrate	20·10	19·32	48·99	20·17
(7) Deviation in :				
(*i*) Benzyl alcohol %	+3·05	+4·54	+5·38	+8·15
(*ii*) Phenyl ethyl alcohol %	−3·42	−4·19	−5·36	−8·24
(*iii*) Benzyl butyrate %	+0·35	−0·34	−0·12	+0·06
(8) Total alcohol %				
(*i*) Found	79·88	80·68	50·93	79·80
(*ii*) Deviation	−0·37	+0·35	+0·02	−0·09
(9) After acetylation				
(*i*) Acid value	3·90	2·60	1·36	1·41
(*ii*) Ester value (Mean)	365·5	363·5	344·6	353·4
(10) Total alcohol % as*				
(*i*) Benzyl alcohol	80·19	80·16	49·44	75·98
(*ii*) Phenyl ethyl alcohol	90·60	90·56	55·86	85·84

*Based on ester value of the mixture after acetylation.

TABLE III

Phenyl ethyl alcohol, Phenyl propyl alcohol and Phenyl ethyl acetate System

Mixtures	I	II	III	IV
(1) % Composition				
(*i*) Phenyl ethyl alcohol	4·34	19·59	39·37	60·77
(*ii*) Phenyl propyl alcohol	40·11	31·51	19·89	8·94
(*iii*) Phenyl ethyl acetate	55·55	48·89	40·74	30·29
(2) Sp. gr. 30°/30°C	1·01797	1·01898	1·02054	1·02156
(3) Density 30°/4°C	1·01499	1·01600	1·01755	1·01859
(4) Sp. vol. 30°/4°C	0·985230	0·984251	0·982750	0·981752
(5) Before acetylation				
(*i*) Acid value	0	0	0	0
(*ii*) Ester value (Mean)	189·9	168·0	139·7	105·2
(6) Found %				
(*i*) Phenyl ethyl alcohol	0·24	16·08	37·77	59·89
(*ii*) Phenyl propyl alcohol	44·21	34·84	21·33	9·49
(*iii*) Phenyl ethyl acetate	55·55	49·14	40·89	30·74
(7) Deviation in :				
(*i*) Phenyl ethyl alcohol%	–4·10	–3·51	–1·60	–0·88
(*ii*) Phenyl propyl alcohol%	+4·10	+3·33	+1·44	+0·46
(*iii*) Phenyl ethyl acetate%	0	–0·25	–0·15	–0·45
(8) Total alcohol %				
(*i*) Found	44·45	50·92	59·10	69·29
(*ii*) Deviation	0	–0·18	–0·16	–0·42
(9) After acetylation				
(*i*) Acid value	0	0	0	0
(*ii*) Ester value (Mean)	329·9	332·2	336·6	340·9
(10) Total alcohol % as*				
(*i*) Phenyl ethyl alcohol	40·51	47·62	57·32	68·94
(*ii*) Phenyl propyl alcohol	45·17	53·11	63·90	76·86

*Based on ester value of the mixture after acetylation.

DISCUSSION AND CONCLUSION

In the case of system 1 consisting of benzyl alcohol, phenyl ethyl alcohol and benzyl butyrate, difference in densities between benzyl alcohol and phenyl ethyl alcohol, phenyl ethyl alcohol and benzyl butyrate, benzyl alcohol and benzyl butyrate is 0·01870, 0·01786 and 0·03656 respectively. Therefore, the key constituent in this system is phenyl ethyl alcohol. Table II indicates that the deviation in benzyl alcohol and phenyl ethyl alcohol percent increases as phenyl ethyl alcohol per cent in the mixture rises.

In the case of system I, the deviation in individual alcohol per cent ranges from 3·05 to 8·24 while the deviation in total alcohol percent ranges from 0·02 to 0·37. But the deviation in total alcohol content as benzyl alcohol and as phenyl ethyl alcohol varies from 0·06 to 3·91 and 4·95 to 10·35 respectively.

In the case of system 2 consisting of phenyl ethyl alcohol, phenyl propyl alcohol and phenyl ethyl acetate, difference in densities between phenyl ethyl alcohol and phenyl propyl alcohol, phenyl ethyl alcohol and phenyl ethyl acetate, and phenyl propyl alcohol and phenyl ethyl acetate is 0·01859, 0·01223 and 0·03082 respectively. The key constituent in this system is phenyl ethyl alcohol. Table III indicates that the deviation in phenyl propyl alcohol and phenyl ethyl alcohol contents decreases as phenyl ethyl alcohol content in the mixture increases.

In the case of system 2, the deviation in individual alcohol percent ranges from 0·46 to 4·1, while the deviation in total alcohol percent ranges from 0·0 to 0·42. But the deviation in total alcohol percent as phenyl ethyl alcohol and phenyl propyl alcohol varies from 0·77 to 3·94 and 0·72 to 7·15 respectively.

ACKNOWLEDGMENTS

The authors wish to convey their thanks to Dr. J. B. Lal, for guidance and one of us (R. N. L.) to C. S. I. R., New Delhi for grant of a Junior Research Fellowship.

REFERENCES

1. Kumar, S., Lal, J. B., and Lal, R. N. *J. and Proc. Oil Tech. Assoc. India*, 15 : Part 1, 11, (1959).
2. Lal, R. N., Patwardhan, V. M. and Lal, J. B. *J. and Proc. Oil Tech. Assoc. India*, 14 : 25, (1958).
3. Weissberger, A. Physical Methods of Organic Chemistry, Vol. I, p. 72, *Interscience Publishers, Inc., New York*, (1947).
4. Lal, R. N. and Lal, J. B. *Industrie Chimique Belge*, No. 6, p. 673, (1960).
5. Srivastava, R. C., Joshi, M. C. and Lal, J. B. *J. Sci. Soc., H. B. T. I. and A I. S. T.* 5 : 28, (1956).
6. Guenther, E. The Essential Oils, Vol. I, p. 265. D. *Van Nostrand Company, Inc. New York*, (1949).

SOME RELATIONS BETWEEN HANKEL TRANSFORMS AND MEIJERS' BESSEL FUNCTION TRANSFORM

By
G. K. GOYAL

Department of Mathematics, University of Rajasthan, Jaipur

[Received on 6th May, 1964].

ABSTRACT

In this paper, some theorems on Hankel and Meijer transforms have been given in the generalised form and interesting particular cases obtained as corollaries, regarding the above theorems. These theorems are applied to obtain a few integral representations of Confluent and Gauss's hypergeometric function.

1. Introduction.—The object of this paper is to prove four theorems on Hankel and Meijer transforms defined by

$$\phi(p) = \int_0^\infty (pt)^{\frac{1}{2}} J_\nu(pt) f(t)\, dt, \quad p > 0 \qquad \ldots.(1\cdot1)$$

and

$$\psi(p) = \sqrt{\frac{2}{\pi}p} \int_0^\infty (pt)^{\frac{1}{2}} K_\nu(pt) f(t)\, dt, \; R(p) > 0 \text{ respectively} \qquad \ldots.(1\cdot2)$$

and to evaluate certain infinite integrals involving hypergeometric functions etc.

Throughout this note (1 1) and (1·2) will be symbolically represented as

$$\phi(p) \overset{J}{\underset{\nu}{=}} f(t) \text{ and } \psi(p) \overset{K}{\underset{\nu}{=}} f(t)$$

respectively and wherever it occurs, the following symbols will denote the series

$$\triangle(\alpha, n) \equiv \frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots\ldots, \frac{\alpha+n-1}{n}$$

$$\triangle(\alpha\pm\beta, n) \equiv \frac{\alpha+\beta}{n}, \ldots\ldots, \frac{\alpha+\beta+n-1}{n}, \frac{\alpha-\beta}{n}, \frac{\alpha-\beta+1}{n}, \ldots\ldots, \frac{\alpha-\beta+n-1}{n}.$$

2. Theorem 1. If $\phi(p) \overset{J}{\underset{\mu}{=}} f(t), p > 0$

and $\psi(p) \overset{J}{\underset{\nu}{=}} t^\lambda \phi(t^{r/s})$, r and s being positive integers

then

$$\psi(p) = \frac{(2r)^{\lambda+\frac{1}{2}}(2s)^{\frac{1}{2}}}{p^{\lambda+1}} \int_0^\infty G^{s,r}_{2r,\,2s} \left\{ \left(\frac{2r}{p}\right)^{2r} \left(\frac{t}{2s}\right)^{2s} \middle| \begin{matrix} \triangle\left(\mp\frac{\nu}{2} - \frac{\lambda}{2}, r\right) \\ \triangle\left(\pm\frac{\mu}{2}, s\right) \end{matrix} \right\} f(t)dt, \ldots.(2\cdot1)$$

provided that the Hankel transforms of $|f(t)|$, $|t^{\lambda}\ \phi(t^{r/s})|$ exist, (2.1) is absolutely convergent and $R(\lambda) < 0$, $R\left(\lambda+\nu+\frac{3}{2}+\frac{\mu r}{s}+\frac{r}{2s}\right) > 0$.

Proof: The value of $\phi\ (x^{r/s})$ is obtained from (1·1) and substituted in

$$\psi(p) = \int_0^\infty (px)^{\frac{1}{2}} J_\nu\ (px)\ x^{\lambda}\ \phi(x^{r/s})\ dx,$$

we get on changing the order of integration in the above

$$\psi(p) = p^{\frac{1}{2}} \int_0^\infty t^{\frac{1}{2}} f(t)\ dt \int_0^\infty x^{\lambda+\frac{r}{2s}+\frac{1}{2}} J_\mu\ (tx^{r/s})\ J_\nu\ (px)\ dx$$

$$= \frac{2^{\lambda+\frac{r}{2s}+\frac{1}{2}}}{p^{\lambda+\frac{r}{2s}+1}} \int_0^\infty t^{\frac{1}{2}} f(t)\ dt \int_0^\infty y^{\frac{\lambda}{2}+\frac{r}{4s}-\frac{1}{4}}\ G^{1\ 0}_{0\ 2}\left(y \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

$$\times\ G^{1\ 0}_{0\ 2}\left(\frac{4^{r/s-1}}{p^{2r/s}}\ t^2\ y^{r/s} \,\middle|\, \frac{\mu}{2}, -\frac{\mu}{2}\right) dy$$

as [3, p. 219]
$$J_\nu\ (x) = G^{1\ 0}_{0\ 2}\left(\frac{x^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right) \qquad \ldots.\ (2{\cdot}2)$$

Now evaluating the y-integral by the result given by Saxena [4], we arrive at the required result on using [3, p. 209].

$$x^{\sigma}\ G^{m\ n}_{p\ q}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{m\ n}_{p\ q}\left(x \,\middle|\, \begin{matrix} a_r+\sigma \\ b_s+\sigma \end{matrix}\right) \qquad \ldots.\ (2{\cdot}3)$$

The change of the order of integration is justified under the conditions stated with the theorem.

Now one of the corollaries of the above theorem, obtained by giving special values to r and s, is mentioned below.

Corollary. If
$$\phi\ (p) \overset{J}{\underset{\nu}{}}\ f(t)$$

and
$$\psi\ (p) \overset{J}{\underset{2\nu}{}}\ t^{-\frac{1}{2}}\ \phi\ (t^2)$$

then
$$2^{\frac{1}{2}}\ p^{\frac{1}{4}}\ \psi(2\sqrt{p}) \overset{J}{\underset{\nu}{}}\ t^{-2}\ f\left(\frac{1}{t}\right) \qquad \ldots.\ (2{\cdot}4)$$

where $R(\nu) > -\frac{1}{2}$, $p>0$, and that the Hankel transforms of $|\ f(t)\ |$, $|\ t^{-1/2}\ \phi(t^2)\ |$ and $|\ t^{-2}\ f(\frac{1}{t})\ |$ exist.

This is readily obtained on taking $r=2$, $s=1$, $\lambda = -\frac{1}{2}$, $\mu = \nu$ and replacing ν by 2ν.

Example 1. Let $f(t) = t^{\frac{1}{2}-\nu}(t^2+a^2)^{-\frac{1}{2}}[(t^2+a^2)^{\frac{1}{2}} - a]^{\nu}$

then [**2**, p. 26] $\phi(p) = p^{-\frac{1}{2}} e^{-ap}$

where $R(a) > 0$, $R(\nu) > -1$, $p > 0$,

and [**2**, p. 30]

$$\psi(p) = \frac{p^{2\nu+\frac{1}{2}} \Gamma\nu}{2^{2\nu+1} a^{\nu} \Gamma(2\nu+1)} {}_1F_1(\nu\,;\,2\nu+1\,;-p^2/4a)$$

where $R(\nu) > 0$, $R(a) > 0$, $p > 0$.

Applying (2·4) we obtain,

$$\frac{p^{\nu+\frac{1}{2}}\Gamma\nu}{a^{\nu}\,\Gamma(2\nu+1)}\,{}_1F_1\left({\nu \atop 2\nu+1}\,;-\frac{p}{a}\right) \underset{\nu}{\overset{J}{=}} t^{\nu-3/2}\left[\frac{1}{t^2}+a^2\right]^{-\frac{1}{2}}\left[\left(\frac{1}{t^2}+a^2\right)^{\frac{1}{2}}-a\right]^{\nu} \quad \ldots . (2\cdot5)$$

where $1 < R(\nu) < \frac{5}{2}$.

3. Theorem 2. If $\phi(p) \underset{\mu}{\overset{J}{=}} f(t), p > 0$

and $\psi(p) \underset{\nu}{\overset{J}{=}} t^{\lambda}\phi(t^{-r/s})$, r and s being positive integers

then

$$\frac{p^{\lambda+1}}{(2r)^{\lambda+\frac{1}{2}}(2s)^{\frac{1}{2}}}\psi(p) = \int_0^{\infty} G^{s+r,0}_{0,2s+2r}\left\{\left(\frac{p}{2r}\right)^{2r}\left(\frac{t}{2s}\right)^{2s} \middle| \triangle\left(\frac{\mu}{2}, s\right), \triangle\left(\frac{\pm\nu+\lambda}{2}, r\right), \triangle\left(-\frac{\mu}{2}, s\right)\right\} f(t)\,dt, \quad \ldots . (3\cdot1)$$

provided that the Hankel transforms of $|f(t)|$, $|t^{\lambda}\phi(t^{-r/s})|$ exist (3·1), is absolutely convergent and

$$R\left(\lambda - \frac{r}{2s} - \frac{\mu r}{s} + 1\right) < 0,\ R\left(\lambda + \nu - \frac{r}{2s} + 1\right) > 0.$$

The proof is similar to that of the first theorem and it can be proved if we use [**3**, p. 209]

$$G^{m\,n}_{p\,q}\left(x \middle| {a_r \atop b_s}\right) = G^{n\,m}_{q\,p}\left(x^{-1} \middle| {1-b_s \atop 1-a_r}\right) \quad \ldots . (3\cdot2)$$

along with the results (2·2), (2·3) and the result of Saxena, R. K. [**4**].

Corollary. If $\phi(p) \underset{\nu}{\overset{J}{=}} f(t)$

$$\psi(p) \underset{\nu}{\overset{J}{=}} \phi\left(\frac{1}{t}\right)$$

then $$\frac{1}{4p^{3/2}}\,\psi\,(p^2/4)\;\frac{J}{2\nu}\;t^{3/2} f\,(t^2) \qquad \ldots.(3\cdot3)$$

provided that the Hankel transforms of $|f(t)|$, $\left|\phi\left(\frac{1}{t}\right)\right|$ exist, (3·3) is absolutely convergent and $R(\nu) > -\frac{1}{2}$, $p > 0$.

This is obtained on putting $r=1$, $s=1$, $\lambda=0$ and $\mu=\nu$.

Example 1. Let $f(t) = t^{\lambda}\, e^{-a^2t^2/4} k_{,\mu}\left(\frac{a^2t^2}{4}\right)$

then [2, p. 69]

$$\phi\,(p) = \frac{\pi^{\frac{1}{2}}\,2^{\lambda+\frac{1}{2}}}{p^{\lambda+1}}\, G^{1\;2}_{2\;3}\left(\frac{p^2}{2a^2}\,\middle|\,\begin{matrix}1-\mu,\;1+\mu\\ \frac{3}{4}+\frac{\lambda+\nu}{2},\;\frac{1}{2},\;\frac{3}{4}+\frac{\lambda-\nu}{2}\end{matrix}\right)$$

where $R(\lambda+\nu\pm2\nu) > -\frac{3}{2}$, $|\arg. a| < \frac{\pi}{4}$, $p > 0$

and [2, p. 91]

$$\psi\,(p) = \frac{\pi^{\frac{1}{2}}}{4}\left(\frac{4}{p}\right)^{\lambda+2} G^{2\;2}_{5\;2}\left(\frac{8a^2}{p^2}\,\middle|\,\begin{matrix}\frac{3}{4}-\frac{\lambda+\nu}{2},\;\frac{1}{4}-\frac{\lambda+\nu}{2},\;\frac{1}{2},\;\frac{1}{4}-\frac{\lambda-\nu}{2},\;\frac{3}{4}-\frac{\lambda-\nu}{2}\\ \mu,\;-\mu\end{matrix}\right)$$

where $|\arg a| < \frac{\pi}{4}$, $R(\lambda-\nu) < 1$, $R(\lambda\pm\mu+\frac{\nu}{2}) > 0$, $p > 0$.

Applying (3·3) we obtain,

$$\frac{\pi^{\frac{1}{2}}}{4}\left(\frac{4}{p^2}\right)^{2\lambda+4} G^{2\;2}_{5\;2}\left\{\frac{128a^2}{p^4}\,\middle|\,\begin{matrix}\frac{-1}{4}-\frac{\nu+\lambda}{2},\;\frac{1}{4}-\frac{\lambda+\nu}{2},\;-\frac{1}{2},\;\frac{1}{4}-\frac{\lambda-\nu}{2},\;\frac{-1}{4}-\frac{\lambda-\nu}{2}\\ \mu,\;\mu\end{matrix}\right\}\frac{J}{2\nu}\;t^{2\lambda+\frac{3}{2}}\,e^{-\frac{a^2t^2}{4}}\,K_{,\mu}\left(\frac{a^2t^2}{4}\right), \qquad \ldots.(3\cdot4)$$

where $R(\lambda+\nu\pm2\mu+\frac{3}{2}) > 0$, $R(p) > 0$, $|\arg a| < \pi/4$

4. Theorem 3. If $\phi\,(p)\,\frac{k}{\mu}\,f\,(t)$, $R\,(p) > 0$

and $\psi\,(p)\frac{k}{\nu}\;t^{\lambda}\,\phi\,(t^{-r/s})$, r and s being positive integers

then

$$\frac{p^{\lambda}\,(2\pi)^{r+s-1}}{2^{\lambda+2}\,r^{\lambda+\frac{1}{2}}\,s^{3/2}}\,\psi\,(p) = \int_0^{\infty} G^{\,2s+2r,0}_{\,0,2s+2r}\left\{\left(\frac{p}{2r}\right)^{2r}\left(\frac{t}{2s}\right)^{2s}\,\middle|\,\wedge\left(\frac{\mu}{2}+\frac{3}{4},s\right),\right.$$
$$\left.\wedge\left(\frac{\pm\nu+\lambda}{2}+\frac{3}{4},r\right),\wedge\left(-\frac{\mu}{2}+\frac{3}{4},s\right)\right\}\frac{f\,(t)}{t}\,dt, \qquad \ldots.(4\cdot1)$$

provided that the Meijer transforms of $|f(t)|$, $|t^{\lambda}\phi(t^{-r/s})|$ exist and (4·1) is absolutely convergent

Proof. Use (1·2) to get the value of $\phi(t^{-r/s})$ and substitute in

$$\psi(p) = \sqrt{\frac{2}{\pi}}\, p \int_0^{\infty} (px)^{\frac{1}{2}}\; k_{\nu}(px)\; x^{\lambda}\; \phi(x^{-r/s})\, dx,$$

we get on changing the order of integration in the above

$$\psi(p) = \frac{2}{\pi}\, p^{3/2} \int_0^{\infty} t^{\frac{1}{2}}\; f(t) \left\{ \int_0^{\infty} x^{\lambda - \frac{3r}{2s} + \frac{1}{2}}\; k_{\mu}(tx^{-r/s})\, k_{\nu}(px)\, dx \right\} dt,$$

$$= \frac{1}{\pi} \frac{2^{\lambda - \frac{3r}{2s} - \frac{1}{2}}}{p^{\lambda - \frac{3r}{2s}}} \int_0^{\infty} t^{\frac{1}{2}} f(t)\, dt \int_0^{\infty} y^{\frac{\lambda}{2} - \frac{3r}{2s} - \frac{1}{4}}\; G^{2\,0}_{0\,2}\left(y \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2} \right)$$

$$G^{0\,2}_{2\,0}\left\{ \left(\frac{2}{p}\right)^{\frac{2r}{s}} \frac{4}{t^2}\, y^{r/s} \,\middle|\, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \right\} dy$$

as [3, p. 219]
$$K_{\mu}(x) = \tfrac{1}{2}\, G^{2\,0}_{0\,2}\left(\frac{x^2}{4} \,\middle|\, \frac{\mu}{2}, -\frac{\mu}{2} \right) \qquad \ldots .\ (4·2)$$

Now integrating the y-integral with the help of the result given by Saxena [4], we arrive at the required result after a little simplification.

The change of the order of integration is justified under the conditions mentioned in the theorem.

Corollary. If
$$\phi(p) \underset{\nu}{\overset{k}{=}} f(t) \quad , R(p) > 0$$

and
$$\psi(p) \underset{\nu}{\overset{k}{=}} \frac{1}{t}\; \phi\left(\frac{1}{t}\right)$$

then
$$\sqrt{\frac{8}{\pi p^3}}\; \psi(p^2/4) \underset{2\nu}{\overset{k}{=}} t^{-\frac{1}{2}} f(t^2) \qquad \ldots .\ (4·3)$$

provided that the Meijer transforms of $|f(t)|$, $|\frac{1}{t}\phi(\frac{1}{t})|$ and $|t^{-\frac{1}{2}} f(t^2)|$ exist.

This is obtained on taking $r = 1$, $s = 1$, $\mu = \nu$ and $\lambda = -1$.

Similarly we can arrive at the following result on taking $r = 1$, $s = 1$, $\mu = \nu$ and $\lambda = 1$ in the same theorem.

If
$$\phi(p) \underset{\nu}{\overset{k}{=}} f(t) \quad , R(p) > 0$$

and
$$\psi(p) \overset{k}{\underset{\nu}{=}} t\,\phi\left(\frac{1}{t}\right)$$

then
$$\sqrt{\frac{p}{2\pi}}\,\psi\,(p^2/_4) \overset{k}{\underset{2\nu}{=}} t^{3/2} f\,(t^2) \qquad \ldots\ldots (4\cdot4)$$

provided that the Meijer transforms of $|\, f\,(t)$, $|\, t\,\phi\left(\frac{1}{t}\right)|$ and $|\, t^{3/2} f\,(t^2)\,|$ exist.

Also that if we put $r=s=1$ and $\lambda = -\ \sigma - \frac{1}{2}$ in the above theorem, it is reduced to a result given by C. B. L. Varma [6].

Example 1. Let $(t) = t^{\nu+\frac{1}{2}} I_{2\nu}(a\,t^{1/2})\, J_{2\nu}(at^{1/2})$

then [2, p. 148] $\phi\,(p) = 2^{-\nu-\frac{1}{2}}\, a^{2\nu+1}\, p^{-2\nu-1}\, J_{\nu-\frac{1}{2}}\,(a^2/_{2p})$

where $R\,(\nu) > -\frac{1}{2}$, $R\,(p) > 0$

and (2, p. 137) $\psi\,(p) = \dfrac{a^{4\nu}}{\sqrt{\pi}}\, p^{-3\nu+\frac{1}{2}}\, \Gamma(2\nu+\frac{1}{2})\, {}_1F_0\left(2\nu+\frac{1}{2}\,;\ -\dfrac{a^4}{4p^2}\right)$

where $R\,(\nu) > -\frac{1}{2}$, $R\,(p) > |\, Im\,(a^2/_2)\,|$

Applying (4·3), we have

$$t^{2\nu+\frac{1}{2}}\, I_{2\nu}\,(at)\, J_{2\nu}\,(at) \overset{k}{\underset{2\nu}{=}} \frac{2^{\frac{3}{2}}\ \Gamma(2\nu+\frac{1}{2})}{\pi p 3\nu+2}\ {}_1F_0\left(2\nu+\frac{1}{2}\,;\ \frac{4a^4}{p^4}\right) \qquad \ldots\ldots (4\cdot5)$$

where $R\,(p) > a$, $R\,(\nu) > -\frac{1}{4}$

5. Theorem 4. If $\phi\,(p) \overset{k}{\underset{\mu}{=}} f\,(t)$, $R\,(p) > 0$

and $\psi\,(p) \overset{k}{\underset{\nu}{=}} t^{\lambda}\,\phi\,(t^{r/s})$, r and s being positive integers

then

$$\frac{p^{\lambda}\,(2\pi)^{r+s-1}}{r^{\lambda+\frac{1}{2}}\,s^{3/2}\,2^{\lambda+2}}\,\psi\,(p) = \int_0^{\infty} G^{2s,\,2r}_{2r,\,2s}\left\{\left(\frac{2r}{p}\right)^{2r}\left(\frac{t}{2s}\right)^{2s}\middle|\ \begin{matrix}\triangle\left(\frac{1}{4}\mp\frac{\nu}{2}-\frac{\lambda}{2},\ r\right)\\ \triangle\left(\pm\frac{\mu}{2},\ s\right)\end{matrix}\right\}\frac{f\,(t)}{t}\,dt, \qquad \ldots\ldots (5\cdot1)$$

provided that the Meijer transforms of $|f(t)|$, $|t^{\lambda} \phi(t^{r/s})|$ exist and (5·1) is absolutely convergent.

The proof is similar to that of the previous theorem and it can be proved on using (3·2) along with (1·2), (4·2) and the result by Saxena [4].

In the above theorem if we put $r = n$ and $s = 1$, we get a result obtained by Sharma [5].

ACKNOWLEDGMENT

The author is highly grateful to Dr. K. C. Sharma of the University of Rajasthan, Jaipur, for his keen interest and guidance in the preparation of this paper.

REFERENCES

1. Erdelyi, *et al.* Tables of Integral transforms, Vol. I, (1954).
2. Erdelyi, *et al* Tables of Intedral transforms, Vol. 2 ,(1954).
3. Erdelyi, *et al.* Higher Trancendental functions, Vol. 1, (1954).
4. Saxena R. K. Some theorems on Generalised Laplace Transform—I, *Proc. Nat. Inst. Sci. India*, 26A, No 4, (1960).
5. Sharma, K. C. Thesis on Generalisation of Laplace Integral, approved for Ph.D. by the University of Rajasthan.
6 Varma, C. B. L. Some properties of K—Transforms involving Meijer G—function. *Proc. Nat. Acad. Sc., India*, 28 : 200-207, (1959).

ON THE DEVELOPMENT OF UNSTEADY BOUNDARY-LAYER THEORY

By

KRISHNA LAL

Lecturer in Mathematics, Engineering College, Banaras Hindu University, Varanasi

[Received on 1st January, 1965]

ABSTRACT

In this paper, I have studied the development of unsteady boundary layer theory when the velocity potential of flow are of the forms

$$U(x, t) = U_0(x)\, e^{\pm nt}.$$

It is based on the work of Lord Rayleigh (2) and Schlichting (3) for the oscillatory non-steady boundary-layer. The equations of non-steady boundary layer, have been used to discuss the development of boundary-layer theory for the persent type of flow. In § 3 interesting result regarding the velocity at the outer edge of the boundary-layer has been obtained. It is shown that a potential flow which is an exponential function of time is not inducing a steady, secondary motion at a large distance from the wall as a result of viscous forces.

§ 1. Formulation of the Problem

For two-dimensional non-steady boundary-layer, equations of motion (1) are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \ldots\ldots \quad 1{\cdot}1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots\ldots \quad 1{\cdot}2$$

The pressure impressed on the body follows from the non-steady Bernoulli equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}, \qquad \ldots\ldots \quad 1{\cdot}3$$

where $U(x, t)$ stands for the prescribed non-steady potential motion.

The integration of the non-steady boundary-layer equations from 1·1 to 1·3 can be carried out in most cases by a process of successive approximations. In the first-instant after the motion had started from rest, the boundary-layer is very thin and the viscuos term $\nu\frac{\partial^2 u}{\partial y^2}$ in equation 1·1 is very large, whereas the convective terms retain their normal values. The viscous term is then balanced by the non-steady accelaration $\frac{\partial u}{\partial t}$ together with the pressure term in which, at first the contribution of $\frac{\partial U}{\partial t}$ is of major importance. Selecting a system of co-ordinates which is at rest with respect to the body and assuming that the fluid moves with

respect to the body and at rest, we can make the assumption that the velocity is composed of two terms,

$$u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) \qquad \ldots\ldots 1{\cdot}4$$

For the first approximation, u_0 satisfies the differential equation

$$\frac{\partial u_0}{\partial t} - \nu \frac{\partial^2 u_0}{\partial y^2} = \frac{\partial U}{\partial t} \qquad \ldots\ldots 1{\cdot}5$$

with the boundary conditions

$$y = 0 : u_0 = 0 \; ; \; y \to \infty : u_0 \to U(x, t) \qquad \ldots\ldots 1{\cdot}6$$

The equation for the second approximation is

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial y^2} = U \frac{\partial U}{\partial x} - u_0 \frac{\partial u_0}{\partial x} - v_0 \frac{\partial u_0}{\partial y} \qquad \ldots\ldots 1{\cdot}7$$

with the boundary conditions

$$u_1 = 0 \text{ at } y = 0 \; ; \; u_1 \to 0 \text{ at } y \to \infty , \qquad \ldots\ldots 1{\cdot}8$$

which is obtained from equation 1·1 in which convective term is calculated from u_0 and in which convective pressure term has also been considered. Besides the equations 1·5 and 1·7, we have equations of continuity to be considered for u_0, v_0, u_1 and v_1

§ 2

In this section, we find solution for the first approximation, from the equations 1·5 and 1·6.

Suppose that the potential velocity distribution, is given by $U_0(x)$. The potential flow in the case of an exponential fluctuations with a frequency n is then given by

$$U(x, t) = U_0(x)\, e^{nt} \qquad \ldots 2{\cdot}1$$

Now a dimensionless co-ordinate defined by

$$\eta = y \sqrt{\frac{n}{\nu}} \qquad \ldots\ldots 2{\cdot}2$$

is introduced and it is assumed that for the first approximation, stream function ψ^0 has form

$$\psi^0(x, y, t) = \sqrt{\frac{\nu}{n}}\; U_0(x)\, F^0(\eta)\, e^{nt} \qquad \ldots\ldots 2{\cdot}3$$

Substituting from equation 2·3 into equation 1·5, with the definition of ψ^0 given by

$$\frac{\partial \psi^0}{\partial y} = u_0, \quad \frac{\partial \psi^0}{\partial x} = -v_0,$$

we have differential equation of F^0

$$\frac{dF^0}{d\eta} - \frac{d^3 F^0}{d\eta^3} = 1 \qquad \ldots\ldots 2{\cdot}4$$

with the boundary conditions

$$\left.\begin{aligned} \eta = 0 : F^0 = \frac{dF^0}{d\eta} = 0, \\ \eta = \infty : \frac{dF^0}{d\eta} = 1 \end{aligned}\right] \qquad \ldots\ldots \; 2{\cdot}5$$

From the equations 2·4 and 2·5, we get

$$F^0 = \eta - e^{-\eta} - 1 \qquad \ldots\ldots \; 2{\cdot}6$$

Hence,

$$\frac{dF^0}{d\eta} = 1 - e^{-\eta} \qquad \ldots\ldots \; 2{\cdot}7$$

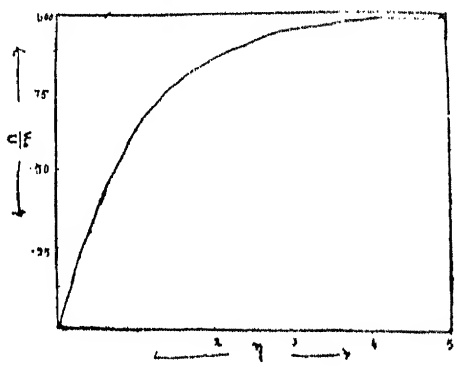
Thus, $u_0 = \frac{\partial\psi^0}{dy} = U\ (x) \left[1 - e^{-\eta} \right]$ .... 2·8

From equation 2·8, the variation of $\frac{u_0}{U}$ with η has been shown graphically.

TABLE

η	0	·1	·2	·3	·4	·5	1·0	1·5	2·0	2·5
$\frac{u_0}{U}$	0	·0952	·1813	·2592	·3997	·3935	·6321	·7769	·8647	·9179
η	3·0	3·5	4·0	4·5	4·6	4·7	4·8	4·9	5·0	∞
$\frac{u_0}{U}$	·9579	·9698	·9817	·9889	·9899	·9909	·9918	·9926	·9933	1

GRAPH



However if we consider the exponentially decreasing function of time for the potential velocity distribution such that $U(x, t) = U_0(x)\, e^{-nt}$ .... 2·9

and for η and ψ^0, take

$$\eta = y\sqrt{\frac{n}{\nu}}, \qquad \text{..... 2·11}$$

$$\psi^0(x, y, t) = \sqrt{\frac{\nu}{n}}\, U_0(x)\, F^0(\eta)\, e^{-nt} \qquad \text{..... 2·12}$$

then we have similar expression as given by equation 2·4 which can now be easily solved.

§ 3

In this section, we find solution for the second approximation, with the help of equations 1·7 and 1·8. Accordingly we take

$$\psi^1(x, y, t) = U_0\frac{dU_0}{dx}\cdot\frac{1}{n}\sqrt{\frac{\nu}{n}}\left[F_1^1(\eta)\, e^{2nt} + F_2^1(\eta)\right] \qquad \text{.... 3·1}$$

where F_1^1 stands for the non-steady component and F_2^1 has been used for the steady component.

Thus, we have for the present case,

$$\left.\begin{aligned} u_0 &= U_0\frac{dF^0}{d\eta}\, e^{nt} \\ v_0 &= -\sqrt{\frac{\nu}{n}}\frac{dU_0}{dx}\, F^0\, e^{nt} \\ u_1(x, y, t) &= U_0\frac{dU_0}{dx}\,\frac{1}{n}\left[\frac{dF_1^1}{d\eta}\, e^{2nt} + \frac{dF_2^1}{d\eta}\right] \end{aligned}\right\} \qquad \text{.... 3·2}$$

Substituting from equations 2·1 and 3·2 into 1·7, we get

$$U_0\frac{dU^0}{dx}\left[\left(2\frac{dF_1^1}{d\eta} - \frac{d^3F_1^1}{d\eta^3}\right)e^{2nt} - \frac{d^3F_2^1}{d\eta^3}\right]$$

$$= U_0\frac{dU_0}{dx}\left[1 - \left(\frac{dF^0}{d\eta}\right)^2 + F^0\left(\frac{d^2F^0}{d\eta^2}\right)\right]e^{2ni} \qquad \text{.... 3·3}$$

Thus on equating the coefficients of e^{2nt} and the terms independent of the time factor, we have

$$2\frac{dF_1^1}{d\eta} - \frac{d^3F_1^1}{d\eta^3} = 1 - \left(\frac{dF^0}{d\eta}\right)^2 + F^0\left(\frac{d^2F^0}{d\eta^2}\right) \qquad \text{.... 3·4}$$

and

$$\frac{d^3F_2^1}{d\eta^3} = 0 \qquad \text{.... 3·5}$$

The boundary conditions are

$$\eta = 0 : F_1^1 = \frac{dF_1^1}{d\eta} = 0 ; \eta = \infty : \frac{dF_1^1}{d\eta} = 0 \quad \ldots . \quad 3{\cdot}6$$

$$\eta = 0 : F_2^1 = \frac{dF_2^1}{d\eta} = 0 ; \eta = \infty : \frac{dF_2^1}{d\eta} = \text{finite} \quad \ldots . \quad 3{\cdot}7$$

Solution of epuation 3·4 with the boundary conditions of 3·6, is

$$F_1^1 = -\frac{1}{\sqrt{2}} e^{-\sqrt{2}\,\eta} - \eta e^{-\eta} + \frac{1}{\sqrt{2}} \quad \ldots . \quad 3{\cdot}8$$

Thus

$$\frac{dF_1^1}{d\eta} = e^{-\sqrt{2}\,\eta} + e^{-\eta} (\eta - 1) \quad \ldots . \quad 3{\cdot}9$$

The above equations are written as

$$F_1^1 = -\frac{1}{\sqrt{2}}\left(F^o - \eta + 1\right)\left[(F^o - \eta + 1)^{\sqrt{2}-1} + \sqrt{2}\,\eta\right] + \frac{1}{\sqrt{2}} \quad \ldots . \quad 3{\cdot}11$$

and

$$\frac{dF_1^1}{d\eta} = \left(1 - \frac{dF^o}{d\eta}\right)\left[\left(1 - \frac{dF^o}{d\eta}\right)^{\sqrt{2}-1} + \eta\right] + \frac{dF^o}{d\eta} - 1 \quad \ldots . \quad 3{\cdot}12$$

which shows that F_1^1 is expressible in terms of η and F^o. On integrating equation 3·5, we have

$$F_2^1 = C_1 \eta^2 + C_2 \eta + C_3 \quad \ldots . \quad 3{\cdot}13$$

where C_1, C_2 and C_3 are constants to be determined from the boundary conditions It is clear from equation 3·13 that a solution satisfying the equations of set 3·7 can not be found. Hence the possible conditions are

$$F_2^1 = 0 \text{ and } \frac{dF_2^1}{d\eta} = 0 \text{ at } \eta = \infty$$

Thus we see that in the present case

$$u_1\,(x,\ \infty,\ t) = 0 \quad \ldots . \quad 3{\cdot}14$$

From above discussion, we see that a potential flow which is exponential with the respect to time is not inducing a steady, secondary (streaming) motion at a large distance from the wall due to viscous forces. In the periodic oscillating flow, it has been shown by Lord Rayleigh[2] and Schlichting[3] that the potential flow induces a secondary flow at large distance from the wall as a result of viscous forces of the magnitude

$$u_1\,(x, , \infty\ t) = -\frac{3}{4n}\, U_0 \frac{dU_0}{dx} \quad \ldots . \quad 3{\cdot}15$$

REFERENCES

1. Pai, S. Viscous Flow Theory I—Laminar Flow Chapter XII (I. N. C. 1956).
2. Rayleigh, L. *Phil. Trans.*, p. 175, (1883).
3, Schlichting, H. *Phy. Z.*, 33 : p. 327, (1932).

WAVES IN A HEAVY INCOMPRESSIBLE FLUID OF FINITE DEPTH AND OF VARIABLE DENSITY. I

By

P. D. ARIEL *and* P. K. BHATIA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 1st January, 1965]

ABSTRACT

The problem of waves in a heavy incompressible viscous fluid of variable density in the presence of a uniform magnetic field $\vec{H_0}$ in a direction paralled to z—axis, *i.e.* parallel to the force of gravity, has been studied before (Hide, 1955), Hide studied the case when the fluid is of finite depth and the bounding surfaces are both free.

In the present note we have studied the problem of waves in a stratified viscous incompressible fluid of finite depth confined between two surfaces which are both rigid and ideal conductors. The problem has been considered for the following two cases :

(*i*) Waves in the absence of magnetic field,

(*ii*) Waves in an ideal conductor in the presence of buoyancy forces.

The explicit solutions have been obtained for both the cases.

§ 1. Introduction

When a plane horizontal layer of an incompressible fluid, stratified in the verticle such that its density ρ_0 varies with z, is disturbed initially then the system in the absence of a damping agency such as viscosity, would oscillate about the mean position with a constant amplitude. The motion then is a horizontally propagated gravity wave. Chandrasekhar (1955), Hide (1955*a*) have considered the effect of viscosity on the same problem.

Hide (1955) considered the problem of waves in a heavy incompressible viscous electrically conducting fluid in the presence of a magetic field $\vec{H_0}$which is uniform and is directed in a direction parallel to the force of gravity. He obtained the appropriate perturbation theory for a general density field $\rho_0\ (z)$ and coefficient of viscosity $\mu_0\ (z)$ for a fluid of constant electrical conductivity σ and constant magnetic permeability k. He obtained the solution in terms of integrals and showed that it is characterized by a variational principle. He further considered the case of a fluid of finite depth d stratified in the verticle according to the law

$$\rho_0 = \rho_1 \exp\ (\beta z), \qquad \ldots. \quad (1)$$

where ρ_1 and β are constants. In his study he assumed that ν, the coefficient of kinematic viscosity, is constant.

Hide has studied the problem for the following three cases when both bounding surfaces of fluid of finite depth are free.

(*i*) Waves in the absence of magnetic field, (Hide, 1955*a*)

(*ii*) Waves in the absence of buoyancy forces, (Hide, 1955)

(*iii*) Waves in an ideal conductor in the presence of buoyancy forces. (Hide, 1955)

In the present note we have considered the same problem (as considered by Hide) when the fluid considered is of finite depth d and is stratified according to the law (1) and the fluid is confined between two boundaries which are rigid and ideal conductors. The study has been carried out for the following two cases,

(1) Waves in the absence of magnetic field,

(2) Waves in an ideal conductor in the presence of buoyancy forces.

We have obtained the explicit solutions for both the cases.

§ 2. Formulation of the problem

The equations governing the motion are [Hide (1955) equation (32)]

$$n \{ k^2 \rho_0 w - D (\rho_0 Dw) \} + \frac{kH_0}{4\pi} (D^2 - k^2) Dh - \left(\frac{gk^2 D\rho_0}{n}\right) w$$
$$+ \mu_0 (D^2 - k^2)^2 w + 2D\mu_0 (D^2 - k^2) Dw + D^2\mu_0 (D^2 + k^2) w = 0, \qquad \ldots\ldots (2)$$

and

$$\{n - \eta (D^2 - k^2)\} h = H_0 Dw, \qquad \ldots\ldots (3)$$

where

ρ_0 denotes the density before the system is disturbed,

w denotes the z–component of the velocity vector $\vec{q}$ at a fixed point,

$\vec{g}$ is the acceleration due to gravity, components $(0, 0, -g)$,

μ_0 is the coefficient of viscosity, assumed to be variable in the undisturbed state,

k is the coefficient of magnetic permeability assumed to be a constant,

$\vec{H_0}$ is the magnetic field diiected towatds the z-axis in the undisturbed state,

h is the z-component of the magnetic field in the perturbed state,

n is the rate at which the system departs from equilibrium,

k is the total wave number of the initial disturbance,

$D \equiv d/dz$ and $\eta = [4\pi K\sigma]^{-1}$

§ 3. Boundary Conditions

The appropriate boundary conditions for the present problem are,

(*i*) $w = Dw = 0$ at rigid boundary $\qquad \ldots\ldots (4)$

(*ii*) $h = 0$ at a surface bounded by an ideal conductor. $\qquad \ldots\ldots (5)$

Let w_i and h_i be solutions of equations (2) and (3) belonging to the characteristic value n_i, and w_j and h_j be the solutions belonging to n_j.

If we multiply equation (2) for i by w_j and integrate over the whole verticle exetnt of the fluid (denoted by L) and multiply equation (3) for j by $(k/4\pi)$ $(D^2 - k^2)\, h_i$ and integrate, we obtain, on combining the two results [Hide (1955) equation (54)].

$$n \left\{ I_1 + I_4 + I_5 - \frac{g}{n^2} I_2 \right\} + I_3 + \eta k^2 \{ I_4 + 2I_8 + I_6 \} = 0, \qquad \ldots\ldots (6)$$

where

$$I_1 = \int_L \rho_0 \{ w^2 + \frac{1}{k^2} (Dw)^2 \} dz, \quad \dots\dots (7)$$

$$I_2 = \int_L w^2 D\rho_0 \, dz, \quad \dots\dots (8)$$

$$I_3 = \int_L \mu_0 \{k^2 w^2 + 2 (Dw)^2 + \frac{1}{k^2} (D^2 w)^2\} dz + \int_L D^2 \mu_0 \, w^2 \, dz, \quad \dots\dots (9)$$

$$I_4 = \frac{k}{4\pi} \int_L h^2 \, dz, \quad \dots\dots (10)$$

$$I_5 = \frac{K}{4\pi k^2} \int_L (Dh)^2 \, dz, \quad \dots\dots (11)$$

$$I_6 = \frac{K}{4\pi k^4} \int_L (D^2 h)^2 \, dz. \quad \dots\dots (12)$$

§ 4. A Continuously stratified fluid of finite depth

We consider the problem of a continuously stratified fluid of depth d in which the undisturbed density follows the law (1) and $\mu_0(z)$ is given by

$$\mu_0(z) = \nu \rho_1 \exp(\beta z), \quad \dots\dots (13)$$

being the coefficient of kinematic viscosity, assumed to be constant. In order that the density variation does not compare with the average density, we make a furthher assumption that

$$|\beta d| << 1, \quad \dots\dots (14)$$

In order that the boundary conditions

$$w = Dw = 0 \qquad \text{at } z = 0 \text{ and } z = d$$

and

$$h = 0 \qquad \text{at } z = 0 \text{ and } z = d$$

are satisfied let us assume trial functions for $w(z)$ and $h(z)$ as

$$w(z) = A(1 - \cos lz)$$

$$h(z) = B \sin lz \quad \dots\dots (15)$$

Boundary conditions (4) and (5) give

$$l = 2\pi s/d, \quad \dots\dots (16)$$

where s is an integer.

Substituting (15) in (3), we have

$$\{ \eta (l^2 + k^2) + n \} B = H_0 lA. \quad \dots\dots (17)$$

Putting the values of w and h in equations (7) — (12), we obtain

$$I_1 = \tfrac{1}{2} \rho_1 A^2 d (3k^2 + l^2)/k^2, \quad \dots\dots (18)$$

$$I_2 = \tfrac{3}{2} \rho_1 \beta A^2 d, \quad \dots\dots (19)$$

$$I_3 = \rho_1 \nu A^2 d (3k^4 + 2k^2 l^2 + l^4)/2k^2, \quad \dots\dots (20)$$

$$I_4 = K B^2 d/8\pi, \quad \ldots\ldots (21)$$
$$I_5 = K B^2 dl^2/8\pi k^2, \quad \ldots\ldots (22)$$
$$I_6 = K B^2 dl^2/8\pi k^4, \quad \ldots\ldots (23)$$
wherein we have made use of the assumption (14).

Substituting the values of I's in equation (6),

we get
$$n^2 (l^2 + 3k^2) - 3g\,\beta k^2 + \nu n (3k^4 + 2k^2 l^2 + l^4) + \frac{KH_0^2}{4\pi\rho_1} \left\{ \frac{nl^2}{\eta (l^2 + k^2) + n} \right\} (l^2 + k^2) = 0. \quad \ldots\ldots (24)$$

Equation (24) is the basic equation of the problem.

If y denotes the dimensionless growth rate and x is the dimensionless wave number, (when n and k are measured in suitable units) such that x and y are given by
$$x = \frac{kd}{\pi s}, \; y = n \frac{2d^2}{\nu\pi^2 s^2}, \quad \ldots\ldots (25)$$
then equation (24) becomes
$$y^2 + \frac{2\,(16 + 8x^2 + 3x^4)}{4 + 3x^2}\, y - \frac{12Gx^2}{4 + 3x^2} + \frac{4Qy}{y + 2P\,(4 + x^2)} \times \frac{4 + x^2}{4 + 3x^2} = 0, \quad \ldots\ldots (26)$$

where
$$G \equiv \frac{g\beta d^4}{\pi^4 \nu^2 s^2} \quad \ldots\ldots (27)$$
$$Q \equiv \frac{KH_0^2 d^2}{\pi^3 \rho_1 \nu^2 s^2}, \quad \ldots\ldots (28)$$
and
$$P \equiv \frac{\eta}{\nu}, \quad \ldots\ldots (29)$$

Hide (1955), while discussing the similar problem with two free surfaces, has pointed out that the above case can not be discussed with facility when $\nu \to 0$. Following Hide (1955), if new measures for x and y are
$$x = \frac{kd}{\pi s}, \; y = n \left(\frac{d}{\pi s}\right) \left(\frac{4\pi\rho_1}{KH_0^2}\right)^{\frac{1}{2}}, \quad \ldots\ldots (30)$$
then the equation (24) takes the form
$$y^2 + 2yS \left(\frac{16 + 8x^2 + 3x^4}{4 + 3x^2}\right) - \frac{3Bx^2}{4 + 3x^2} + \frac{4y}{y + 2R\,(4 + x^2)} \times \frac{4 + x^2}{4 + 3x^2} = 0, \quad \ldots\ldots (31)$$

where the three new dimensionless paramerters R, B and S are defined as follows :

$$R \equiv \tfrac{1}{2}\eta \left(\frac{\pi s}{d}\right) \frac{1}{V_A}, \qquad \ldots . \quad (32)$$

$$B \equiv \frac{g\beta d^2}{\pi^2 s^2 V_A{}^2} \qquad \ldots . \quad (33)$$

$$S \equiv \tfrac{1}{2}\nu \left(\frac{\pi s}{d}\right) \frac{1}{V_A}, \qquad \ldots . \quad (34)$$

where
$$V_A{}^2 = \frac{K H_0{}^2}{4\pi\rho_1}. \qquad \ldots . \quad (35)$$

The study of this problem based on parameters R, B and S has been carried out in a separate paper by one of the authers (*P. D. A.*) (Ref. 7).

At this point we may point out that when $V_A = 0$ *i. e.* $H_0 = 0$, the equation (31) can not be discussed properly. However, this case ($H_0 = 0$) can be conveniently discussed in terms of parameters G, P and Q *i.e.* by using equation (26).

We, therefore, have carried out the discussion based on equation (26).

In equation (26), G has a form of Grashoff number [Goldstein (1938)] which to some extent determines the relative importance of buoyancy forces and viscous forces, P measures the relative amounts of damping due to the electrical resistance and viscosity. Q is a suitable measure of the magnetic field strength.

Equation (26) is a cubic in y, but can be reduced to quadratic in the followiag three cases (I) $Q = 0$ or $P \to \infty$ (II) $P = 0$ and (III) $G = 0$.

We consider the first two cases in the paper. Case (III) will be discussed in a latter paper.

§ 5. Case I. Waves in the absence of Magnetic field or Waves in a non-conducting fluid

This is the case when either $Q = 0$ or $P \to \infty$ in equation (26). Equation (26) then becomes

$$y^2 + 2\,\frac{16 + 8x^2 + 3x^4}{4 + 3x^2}\, y - \frac{12Gx^2}{4 + 3x^2} = 0. \qquad \ldots . \quad (36)$$

The solution of equation (36) is

$$y = -\frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \pm \left\{ \left(\frac{16 + 8x^2 + 3x^4}{4 + 3x^2}\right)^2 + \frac{12Gx^2}{4 + 3x^2} \right\}^{\frac{1}{2}} \qquad \ldots . \quad (37)$$

Here two cases arise namely (*i*) $G > 0$ (*ii*) $G < 0$.

(*i*) *Unstable stratrification :* $\beta > 0$, $G > 0$.

The value of y corresponding to the positive sign is real and positive. In this case the disturbance grows aperiodically with time, therefore the equilibrium is unstable. The mode of maximum instability is given by the following pair of epuations.

$$(3x_p^4 - 16)(9x_p^4 + 24x_p^2 - 16) - 48\,G = 0 \qquad \ldots. \quad (38)$$

$$y_p^2(4 + 3x_p^2) + 2(16 + 8x_p^2 + 3x_p^4)\,y_p - 12\,Gx_p^2 = 0. \qquad \ldots. \quad (39)$$

(ii) Stable stratification : $\beta < 0$, $G < 0$

In equation (37) y never has a positive real part so that the equilibrium is stable. Whether it is restored periodically or aperiodically depends on the sign of the discriminant.

$$\text{If } f \equiv f(x) = \frac{(16 + 8x^2 + 3x^4)^2}{12x^2(4 + 3x^2)} \qquad \ldots. \quad (40)$$

equation (37) shows that the motion is periodically or aperiodically damped according as

$$G_1 \gtrless f, \qquad \ldots. \quad (41)$$

where $\qquad G_1 = -G$, G_1 being positive.

The Graph 1 gives the values of $f(x)$ against x.

The minimum values 7·8763 of $f(x)$ occurs at $x = 1{\cdot}26605$.

It is easily seen from (41) that motion is oscillatory for those values of x for which line drawn parallel to x - axis at a distance G_1 above it lies above the curve f; otherwise the equilibrium is restored aperiodically.

From Graph 1, we see that

(*i*) when $G_1 \leqslant 7{\cdot}8763$ here $G_1 \leqslant f$ for all values of x, so that all the modes are aperiodically damped.

(*ii*) when $G_1 > 7{\cdot}8763$ there are two non-zero values of x, say x_1 and x_2, for which $G_1 = f$. Oscillations arise only within the wave number range $x_1 < x < x_2$, and aperiodic motion arises elsewhere.

Next we obtain the properties of the motion.

In the case of aperiodical damping there are two damping co-efficients given by

$$-y = \frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \pm \left\{ \left(\frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right)^2 - \frac{12G_1x^2}{4 + 3x^2} \right\}^{\frac{1}{2}} \qquad \ldots. \quad (42)$$

In the case of oscillatory motion there is only one damping co-efficient given by

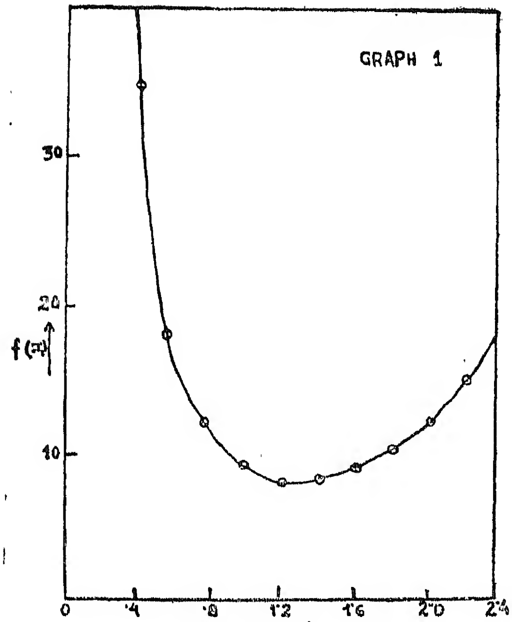
$$-R(y) = \frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \qquad \ldots. \quad (43)$$

The angular frequency of oscillation is given by

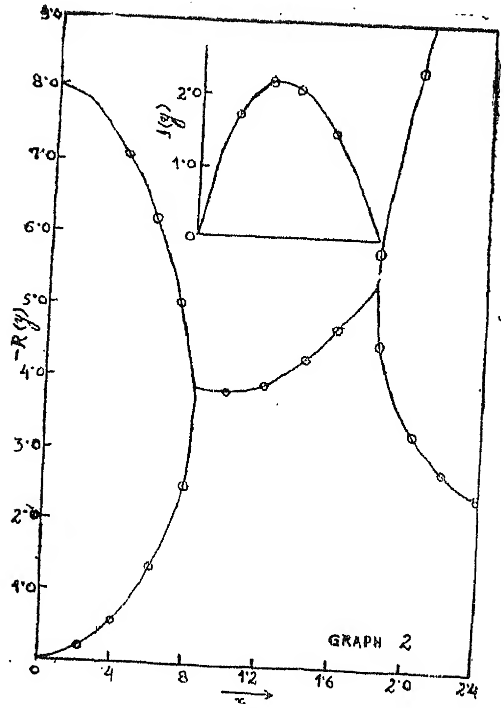
$$I(y) = \pm \left\{ \frac{12G_1x^2}{4 + 3x^2} - \left(\frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right)^2 \right\}^{\frac{1}{2}} \qquad \ldots. \quad (44)$$

These functions are illustrated by Graph 2 for case (*ii*). The corresponding waves and group velocities V_w and V_g satisfy the equations

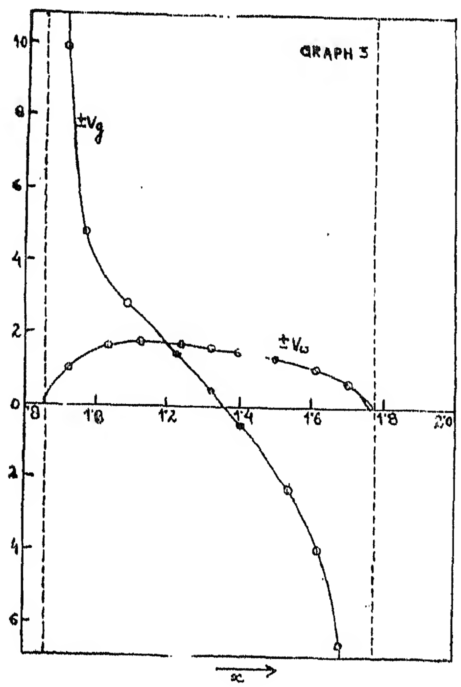
$$V^2_w = \{ I(y) \}^2 / x^2 \qquad \ldots. \quad (45)$$



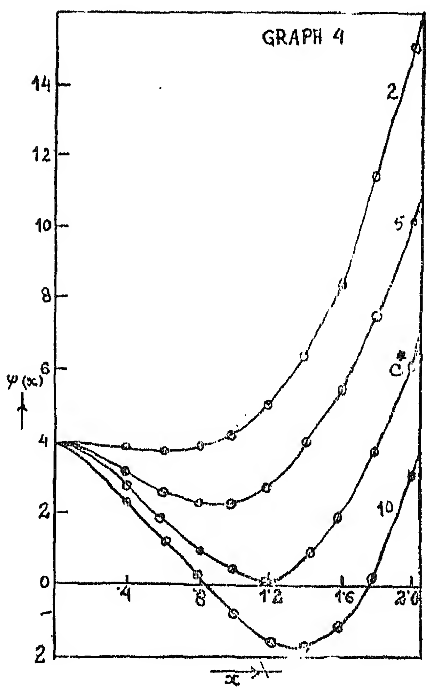
Plot of $f(x)$ against x.



The growth rate y, as a function of the wave number x for $Q = 0$, $G_1 = 10$.



The phase and group velocities Vw and Vg respectively, as functions of the wave number x in case $Q=0$, $G_1 = 10$.



Variation of $\psi(x)$ with x for different values of G_1.

and
$$V^2{}_g = \left\{\frac{d\,I(y)}{dx}\right\}^2$$

$$= \frac{4}{V^2{}_w}\left\{\frac{24G_1}{(4+3x^2)^2} - \frac{(16+8x^2+3x^4)(9x^4+24x^2-16)}{(4+3x^2)^3}\right\}^2 \qquad \ldots\ldots (46)$$

These functions are illustrated by Graph 3 for case (*ii*). It is clearly seen in the Graph that in case (*ii*) the angular frequency $I(y)$ has a maximum value, and from equation (46) we see that the value of x at which the maxima of $I(y)$ occurs, the group velocity vanishes.

§ 6. Case II. Waves in an ideal conductor in the presence of buoyancy forces.

This is the case $P = 0$. For this case equation (26) reduces to

$$y^2 + \frac{2(16+8x^2+3x^4)}{4+3x^2}\,y - \frac{12Gx^2}{4+3x^2} + 4Q\,\frac{4+x^2}{4+3x^2} = 0. \qquad \ldots\ldots (47)$$

Its solution is

$$y = -\frac{16+8x^2+3x^4}{4+3x^2} \pm \left[\left(\frac{16+8x^2+3x^4}{4+3x^2}\right)^2 - 4\left\{\frac{Q(4+x^2)-3Gx^2}{4+3x^2}\right\}\right]^{\frac{1}{2}} \qquad \ldots\ldots (48)$$

Again we have to divide this section into two parts,

(*i*) *Unstable stratification* : $\beta > 0,\ G > 0$.

According to equation (48) when $x > x_0$, x_0 being the real positive root of the equation

$$3Gx_0^2 = Q(4+x_0^2) \qquad \ldots\ldots (49)$$

The value of y corresponding to the positive sign is real and positive. In this case the disturbance grows aperiodically with time, therefore the equilibrium is unstable. There exists one value of x (x_q say) for which the growth rate is maximum and equal to y_q (say).

Differentiating (48) we can show that

$$(9x^4{}_q + 24x^2{}_q - 16)\,[3(Q-3G)\,x^4{}_q + 24Qx^2{}_q + 16(Q+3G)] + 16\,(2Q+3G)^2 = 0 \qquad \ldots\ldots (50)$$

$$y^2{}_q\,(4+3x^2{}_q) + 2\,(16+8x^2{}_q+3x^4{}_q)\,y_q - 12Gx^2{}_q + 4Q\,(4+x^2{}_q) = 0 \qquad \ldots\ldots (51)$$

When $x < x_0$, In equation (47), y no longer has a positive value so that for these values of x, no amplified motion is possible. The motion is periodically or aperiodically damped according as the quantity under the radical sign in equation (48) is negative or positive.

It is seen that the presence of magnetic field gives the motion a stabilizing effect at least for the wave number range $x < x_0$, for which range the motion is unstable in the absence of magnetic field (Case I).

However such motion would not be observed because unstable motion at $x > x_0$ would swamp any other type of motion—the mathematically stable modes of this case of unstable stratification have no physical interest.

(ii) *Stable stratification* : $\beta < 0$, $G < 0$.

From equation (48), we see that y never has a positive real part, so that the equilibrium is stable. Whether it is restored periodicallly or aperiodically depends on the sign of the quantity under the radical sign.

If we put

$$\psi \equiv \psi(x) = \tfrac{1}{4}\,\frac{(16 + 8x^2 + 3x^4)^2}{(4 + 3x^2)\,(4 + x^2)} - \frac{3G_1 x^2}{4 + x^2} \qquad \ldots . \quad (52)$$

where, as before, $G_1 = -G$,

then it follows from equation (48) that the motion is periodically or aperiodically damped according as

$$Q \gtreqless \psi \qquad \ldots . \quad (53)$$

In Graph 4 ψ is plotted as a function of x for different values of G_1. According to criterian (53) we have oscillations for only those values of x for which a line drawn paralled to x–axis at a distance Q above it lies ,above the curve ψ. For all other values of x, the equilibrium is restored aperiodically.

It is seen from Graph 4 that for all values of G_1 (except $G_1 = 0$, a case already discussed – §5) falls monotonically with x from its value $\psi = 4$ at $x = 0$ untill it reaches ψ_{min} $(=\psi_m)$ at x_m. The values of ψ_m and x_m are given by equations (54) and (55).

$$\psi_m = \tfrac{1}{4}\,\frac{(16 + 8x_m^2 + 3x_m^4)^2}{(4 + 3x_m^2)\,(4 + x_m^2)} - \frac{3G_1\, x_m^2}{4 + x_m^2}\,, \qquad \ldots . \quad (54)$$

and

$$27x_m^{10} + 288x_m^8 + 1056x_m^6 + 8x_m^4\,(256 - 27G_1) + 64x_m^2\,(28 - 9G_1) - 384G_1 = 0. \qquad \ldots . \quad (55)$$

From this point on, ψ increases monotonically with x, the curve tending to infinity for larger values of x.

It may be added that x_m has only a single positive real value which satisfies equation (55) as can be verified from Descarte's rule of signs.

From Graph 4 it is easily seen that we are to distinguish the two cases namely $G_1 < G^*$ and $G_1 > G^*$ where G^* is that value of G_1 for which $\psi_{min} = 0$.

The condition $\psi_{min} = 0$ is that the equation $\psi_m = 0$ (*cf.* equation (54)) should have two equal roots *i.e.*

$$9g^3 - (33/16)g^2 - 22g + 8 = 0 \qquad \ldots . \quad (56)$$

where $\qquad g = 3G_1/16 \qquad \ldots . \quad (57)$

Equation (56) is a cubic in G_1. However we chose only that value of G_1 (denoted by G^*) for which x_m given by equation (55) is positive. Thus $G^* = 7{\cdot}8763$.

Here it may be noted that the value of G^* is same as in case 1.

The situation may be summarised as follows :

(*i*) $4 < Q$. There is one value of x, say x'_1, for which $Q = \psi$ Oscillations arise where $0 < x < x'_1$, and aperiodical motion when $x'_1 \leqslant x$.

(*ii*) (*A*) $G_1 \leqslant G^*$

(1) $\psi_m < Q < 4$. There are two non zero values of x say x'_2 and x'_3 for which $Q = \psi$. Oscillations arise within the wave number range $x'_2 < x < x'_3$, and aperiodical motion arisess elsewhere.

(2) $Q \leqslant \psi_m$. In this case $Q \leqslant \psi_m$ for all values of x, so that all modes are aperiodically damped.

(*B*) $G_1 > G^*$

$0 \leqslant Q < 4$. There are two non zero values of x, namely x'_4 and x'_5 for which $Q = \psi$. Oscillations arise within the wave number range $x'_4 < x < x'_5$ and elsewhere the motion is aperiodic.

In table 1 the values of x_m and ψ_m are listed for the different values of G_1.

S. No.	G_1	x_m	ψ_m
1.	1	·4768	3·9188
2.	2	·6821	3·6807
3.	3	·8345	3·3004
4.	4	·9550	2·7982
5.	5	1·0542	2·1920
6.	6	1·1373	1·4435
7.	7	1·2100	·6721
8.	G^*	1·26605	0

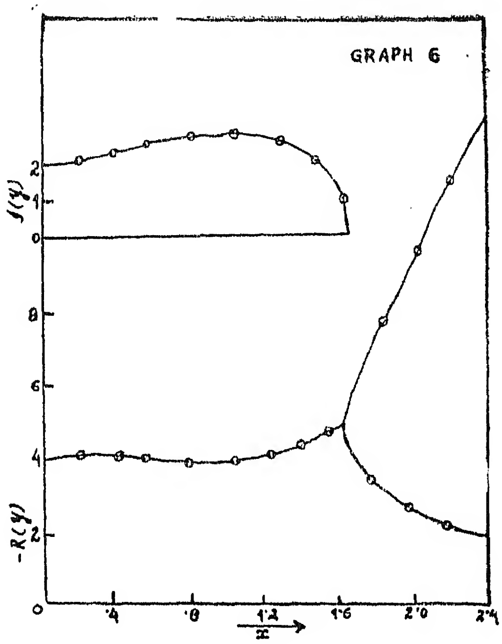
In the case of aperiodic damping there are two damping co-efficients (*see* equation (48)).

$$-y = \frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \pm \left[\left(\frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right)^2 - 4 \left\{ \frac{Q\,(4 + x^2) + 3G_1 x^2}{4 + 3x^2} \right\} \right]^{\frac{1}{2}} \quad \ldots\ldots (58)$$

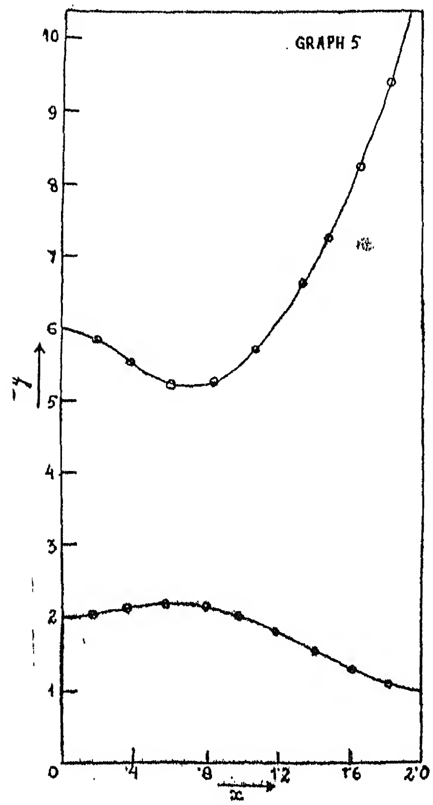
Graph 5 gives the behaviour of $-y$ against x.

For the Oscillatory motion, there is only one damping coefficients given by

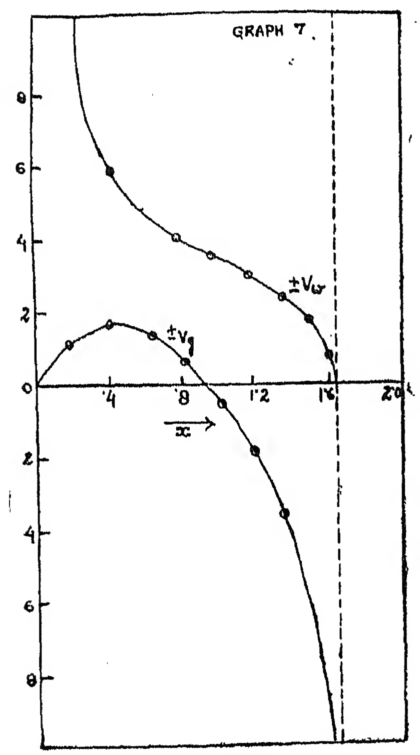
$$-R(y) = \frac{16 + 8x^2 + 3x^4}{4 + 3x^2}. \quad \ldots\ldots (59)$$



The growth rate y, as a function of the wave number x in case $P = 0$, $Q = 5$, $G_1 = 5$.



The growth rate y, as a function of the wave number x in the case $P = 0$, $G_1 = 2$, $Q = 3$.



The phase and group velocities V_w and V_g respectively as function of the wave number x in case $P = 0$, $Q = 5$, $G_1 = 5$.

The angular frequency of oscillation is given by

$$I(y) = \pm \left[4 \left\{ \frac{Q\,(4 + x^2) + 3G_1 x^2}{4 + 3x^2} \right\} - \left(\frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right)^2 \right]^{\frac{1}{2}} \quad \ldots . \quad (60)$$

In Graph 6 these functions are plotted against x for the case (i).

The corresponding wave and group velocities V_w and V_g satisfy the equations

$$V_w^2 = \{ I(y) \}^2 / x^2, \quad \ldots . \quad (61)$$

and

$$V_g^2 = \{ dI(y)/dx \}^2$$

$$= \frac{4}{V_w^2} \left\{ \frac{8\,(2Q - 3G_1)}{(4 + 3x^2)^2} + \frac{(16 + 8x^2 + 3x^4)\,(9x^4 + 24x^2 - 16)}{(4 + 3x^2)^3} \right\}^2 . \quad \ldots . \quad (62)$$

These functions are illustrated by Graph 7 for case (1).

REFERENCES

1. Lamb, H. "Hydrodynamics" Dover Publications, (1945).
2. Lord Rayleigh. Scientific papers, *ii*, 200–7, Cambridge, England, (1900).
3. Goldstein S. Modern developments in fluid dynamics, Oxford (Clarendon Press), Ed., (1938).
4. Chandrasekhar, S. *Proc. Camb. Phil. Soc.* 51 : 162–178, (1955).
5. Hide, R. *Proc. Camb. Phil. Soc.* 51 : 179–201, (1955*a*).
6. Hide, R. *Proc. Camb. Phil. Soc.* (London) 233A : 376–396, (1955).
7. Gandhi, J. M., Ariel, P. D., Swami, L. K. "Waves in a heavy, viscous incompressible, electrically conducting fluid of variable density, confined between two rigid boundaries in the presence of a magnetic field." (To be published), (1964).

RADIAL PULSATIONS OF AN INFINITE CYLINDER IN A MAGNETIC FIELD WITH VARIABLE DENSITY

By

K. M. SRIVASTAVA *and* R. S. KUSHWAHA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 20th February, 1965]

ABSTRACT

We have studied the Radial pulsations of an infinitely conducting infinite cylinder of variable density in presence of a magnetic field which has both toroidal and poloidal Components.

INTRODUCTION

Radial pulsations of an infinitely conducting infinite cylinder has been studied by many researchers[1,2,3,4,5,6]. In all the cases considered by them the change in the total pressure at the boundary becomes zero. Raju and Talwar[7] have studied the pulsations in a Non-force free field with $H\varphi = K.r$ and

$$Hz = K_0 \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}.$$

They had taken the density uniform throughout the cylinder. In this paper we study the effect of density varying according to the law

$$\rho = \rho_c \left(1 - \frac{r^2}{R^2} \right) \tag{1}$$

with $H\varphi$ and Hz as mentioned above. Raju and Talwar have obtained asymptotic solution of the differential equation for higher modes. We have obtained the series solution of differential equation numerically. We have also calculated relative change in magnetic field due to oscillations. We have given requisite graphs showing displacement function ψ against $x = \frac{r}{R}$. We have also plotted frequency (A) against $\frac{H_0^2}{H_s^2}$, where

$$A = \left(\frac{\sigma^2}{\pi G \rho_c} + 4\right) \cdot \frac{1}{\gamma} \tag{2}$$

A graph for the relative change in the magnetic field is also drawn.

FUNDAMENTAL EQUATIONS

The equations governing the equilibrium state for self gravitating infinitely conducting infinite cylinder with variable density are

$$-\frac{1}{\rho}\,\text{grad}\,p - \text{grad}\,V + \frac{1}{4\pi\rho}\,\{\text{curl}\,\overline{H}\} \times H = 0, \tag{3}$$

$$\nabla^2 V = 4\pi G\rho, \tag{4}$$

$$\nabla . \overline{H} = 0, \tag{5}$$

$$\nabla \times \overline{H} = 4\pi \overline{j}, \tag{6}$$

$$\frac{\overline{J}}{\sigma} = \overline{E} = 0. \tag{7}$$

In the following we assume that the magnetic field $H = (0, H\varphi, Hz)$ is dependent upon γ only *i.e.*

$$H\varphi = Kr \tag{8}$$

$$Hz = K_c (1 - x^2)^{\frac{1}{2}} \tag{9}$$

and the density to be variable given by

$$\rho = \rho_c (1 - x^2), \tag{10}$$

with $\quad x = \dfrac{r}{R}$. (11)

In the vacuum

$$\nabla . \overline{H} = 0, \tag{12}$$

and $$\nabla \times H = 0. \tag{13}$$

The only solution admitted by the above is

$$H^v_\varphi = \frac{A}{r}, \tag{14}$$

$$H^v_z = B, \tag{15}$$

where A and B are constants.

In the absence of surface currents

$$B = H^s_z = 0 \tag{16}$$

and $$A = R\,H^s_\varphi = KR^2. \tag{17}$$

Equation of continuity and equation of motion are

$$2\pi\rho r dr = 2\pi\rho_c r_c dr_0,$$

$$i.e.\ \frac{\delta\rho}{\rho_0} = -\frac{1}{r}\frac{\partial}{\partial r}(r\delta r) = -\frac{1}{r}\frac{\partial}{\partial r}(r\xi), \tag{18}$$

and

$$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{dp}{dr} - g + \frac{1}{4\pi\rho}\Big[\{\text{curl}\,\overline{H}\} \times \overline{H}\Big]_r \tag{19}$$

Writing

$$r = r_0 + \xi,\ \rho = \rho_0 + \delta\rho \text{ and } \overline{H} = \overline{H}_0 + \overline{h} \tag{20}$$

the equation governing the radial, pulsation becomes

$$\frac{d^2\xi}{dt^2} = -\frac{1}{\rho}\frac{d\,\delta p}{dr} + \frac{4Gm\xi}{r^2} + \frac{1}{4\pi\rho}\Big[\{\nabla \times \overline{H_0}\} \times \overline{h} + \{\nabla \times \overline{h}\}$$
$$\times \overline{H_0} + \{\nabla \times \overline{h}\} \times \overline{h}\Big]$$

$$\text{or } \rho\left[\frac{d^2\xi}{dt^2} - \frac{4Gm(r)\xi}{r^2}\right] = -\frac{d\,\delta p}{dr} + \frac{1}{4\pi}\Big[\{\nabla \times \overline{H_0}\} \times \overline{h} + \{\nabla \times \overline{h}\}$$
$$\times \overline{H_0} + \{\nabla \times \overline{h}\} \times \overline{h}\Big]. \tag{21}$$

The linearised equations governing the radial pulsations under adiabatic conditions are

$$\rho\left[\frac{d^2\xi}{dt^2} - \frac{4Gm(r)\xi}{r^2}\right] = \frac{d}{dr}\left[\frac{\gamma p}{r}\frac{d}{dr}(r\xi) + H_\varphi^2\frac{d\xi}{dr} + \frac{H_z^2}{4\pi r}\frac{d}{dr}(r\xi)\right] +$$
$$+ \frac{H_\varphi^2}{4\pi r^2}\frac{d}{dr}(r\xi), \tag{22}$$

where

$$\left.\begin{aligned} \overline{h} &= \overline{j}\,\delta H_\varphi + \overline{k}\,\delta H_z \\ \overline{H} &= \overline{j}\,H_\varphi + \overline{k}\,H_z \end{aligned}\right\}, \tag{23}$$

and

$$\left.\begin{aligned} \delta H_\varphi &= -H_\varphi\frac{d\xi}{dr} \\ \delta H_z &= -\frac{H_z}{r}\frac{d}{dr}(r\xi) \end{aligned}\right\}, \tag{24}$$

$$\delta p = -\frac{\gamma p}{r}\frac{d}{dr}(r\xi), \tag{25}$$

$m\,(r)$ = mass of unit length of cylinder of radius r,

$$= \pi\rho_c\, r^2\left(1 - \frac{r^2}{2R^2}\right) \tag{26}$$

and γ denotes the ratio of two specific heats.

In writing down the perturbed equations, the Lagrangian description of the fluid motion is adopted.

The change in the magnetic field in the vacuum is governed by

$$\nabla \cdot \delta\overline{H}^v = 0, \tag{27}$$

$$\nabla \times \delta\overline{H}^v = 0, \tag{28}$$

and

$$\nabla \times \overline{E}^v = -\frac{\partial}{\partial t}(\delta\overline{H}^v) \tag{29}$$

The only possible solution of the above equations is of the form given in (14) and (15). As any finite value of $\delta \bar{H}^{v})$ will give rise to an infinite electric field at infinity, it must vanish identically.

The continuity of pressure at the boundary requires that, neglecting ξ^2 and higher order terms in ξ,

$$-\left[\left(\frac{\gamma p}{r}+\frac{H_z^2}{4\pi r}\right)\frac{d}{dr}(r\xi)+\frac{H_\varphi^2}{4\pi}\frac{d\xi}{dr}\right]_{r=R}=\left[\xi\frac{H_\varphi^v}{4\pi}\frac{d}{dr}(H_\varphi^v)\right]_{r=R}. \tag{30}$$

Assuming the time dependance of all the physical variables as $e^{i\sigma t}$, the equation (22), with the help of (26) and (10), reduces to

$$-\rho_c\left(1-\frac{r^2}{R^2}\right)\left[\sigma^2+4G\pi\rho_c\left(1-\frac{r^2}{2R^2}\right)\xi\right]$$

$$=\frac{d}{dr}\left[\left(\frac{\gamma p}{r}+\frac{H_z^2}{4\pi r^2}\right)\frac{d(r\xi)}{dr}+\frac{H_\varphi^2}{4\pi}\frac{d\xi}{dr}\right]+\frac{H_\varphi^2}{4\pi r^2}\frac{d(r\xi)}{dr}. \tag{31}$$

Substituting the following relations given by (32) to (37)

$$H\varphi = K_r,\ r\leqslant R;\quad Hz=K_0\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}},\ r\leqslant R, \tag{32}$$

$$H_\varphi^v=\frac{KR^2}{r},\ r\geqslant R;\ H_z^v=0,\ r\geqslant R, \tag{33}$$

$$p=P-\frac{2H_s^2x^2+H_0^2(1-x^2)-2H_s^2}{8\pi}, \tag{34}$$

$$P=\frac{1}{12}\pi G\rho_c^2R^2\left(1-\frac{r^2}{R^2}\right)^2\left(5-\frac{2r^2}{R^2}\right), \tag{35}$$

$$p_0=\frac{5}{12}\pi G\rho_c^2R^2-\frac{H_0^2-2H_s^2}{8\pi}, \tag{36}$$

and
$$x=\frac{r}{R},\quad \psi=\frac{\xi}{R}, \tag{37}$$

in equation (31), we get

$$-\frac{\rho_c}{\gamma}(1-x^2)\left[\sigma^2+2\pi G\rho_c(2-x^2)\right]\psi$$

$$=\frac{d}{dx}\left[\frac{1-x^2}{x}(K_1+D)+\frac{B}{x}\right]\frac{d}{dx}(x\psi),$$

which can be written as

$$-(1-x^2)\left[A-\frac{2}{\gamma}x^2\right]\psi=\frac{d}{dx}\left[\frac{(1-x^2)(K_1+D)+B}{x}\right]\frac{d}{dx}(x\psi), \tag{38}$$

with

$$K_1=\frac{(1-x^2)(5-2x^2)}{12}, \tag{39}$$

$$B = \frac{H_s^2}{4\pi^2 G\rho_c^2 R^2 \gamma} \quad ; \quad A = \frac{1}{\gamma}\left[\frac{\sigma^2}{\pi G\rho_c} + 4\right], \tag{40}$$

and

$$D = \frac{1}{8\pi^2 G\rho_c^2 R^2}\left[\left(\frac{2}{\gamma} - 1\right)(H_0^2 - H_s^2) + H_s^2\right], \tag{41}$$

and equation (30) reduces to

$$\frac{d\psi}{dx} = \psi \quad \text{at } x = 1. \tag{42}$$

The subscripts o and s refer to the axis and surface of the cylinder. B and D are always positive for γ lying between 1 and 2. Equation (43) and (44) respectively give the ratios of the frequencies and time periods of pulsations of the n^{th} mode with and without magnetic field,

$$\frac{\sigma_n}{\sigma_{0n}} = \sqrt{\frac{\gamma A_n - 4}{4(n^2\gamma - 1)}}, \tag{43}$$

$$\frac{T}{T_0} = \sqrt{\frac{n^2\gamma - 1}{\gamma A_n - 4}}, \tag{44}$$

where σ_{0n} is the frequency of the n^{th} mode without magnetic field and To is the time period of the n^{th} mode without magnetic field.

The relative change in the magnetic field, by using (24) and (37), can be expressed as

$$\frac{|\delta H|}{H} = \frac{\left[\psi'^2\left\{x^2 + \frac{H_0^2}{H_s^2}(1-x^2)\right\} + \frac{H_0^2}{H_s^2}\frac{\psi^2(1-x^2)}{x^2} + \frac{2H_0^2}{H_s^2}\frac{(1-x^2)}{x}\psi\psi'\right]^{\frac{1}{2}}}{\left[x^2 + \frac{H_0^2}{H_s^2}(1-x^2)\right]^{\frac{1}{2}}}. \tag{45}$$

THE SOLUTION

After simplification, the equation (38) reduces to

$$\begin{aligned} x^2\psi''&\left[\left(B + D + \frac{5}{12}\right) - (1 + D)x^2 + \frac{3}{4}x^4 - \frac{1}{6}x^6\right] \\ + x\psi'&\left[\left(B + D + \frac{5}{12}\right) - 3(1 + D)x^2 + \frac{15}{4}x^4 - \frac{7}{6}x^6\right] \\ + \psi&\left[-\left(B + D + \frac{5}{12}\right) - x^2(1 + D - A) + x^4\left(\frac{21}{20} - A\right) + \frac{11}{30}x^6\right] = 0. \end{aligned} \tag{46}$$

The solution of the above differential equation is sought by the method of series solution.

Let the solution of the equation (46) be

$$\psi = \sum_{\lambda=0}^{\lambda=\infty} a_\lambda x^{\alpha+\lambda}. \tag{47}$$

By substituting this in (46), we find that the indicial equation gives two roots $\alpha = \pm 1$. We therefore expand the solution about $x = o$ neglecting the negative root to avoid the singularity at the axis.

We obtain a four term recursion formula, arbitrarily assuming $a_0 = 1$ (As the equation is linear).

$$a_\lambda \lambda(\lambda + 2) \left(B + D + \frac{5}{12}\right) - a_{\lambda-2} [\lambda^2 (1 + D) - A]$$

$$+ a_{\lambda-4} \left[\frac{-3\lambda^2 - 6\lambda + \frac{24}{5}}{4} - A\right] + \left[\frac{\frac{36}{5} + 4\lambda - \lambda^2}{6}\right] = 0. \tag{48}$$

We obtain the correct value of A from the boundary condition (42) which simplifies to

$$2a_2 + 4a_4 + 6a_6 + 8a_8 + \dots\dots\dots = 0. \tag{49}$$

Firstly we calculate A for different values of B and D *i.e.* H_0 and H_s. Secondly displacement function ψ and relative change in H have been calculated.

The series with proper co-efficients is given by

$$\psi = x - \frac{A - 4 (1 + D)}{8 (B+D+\frac{5}{12})} x^3 +$$

$$+ \left[\frac{\{ 16 (1 + D) - A\} \left\{\frac{4 (1 + D) - A}{8 (B + D + \frac{5}{12})}\right\} - \left(\frac{24}{5} - A\right)}{24 \quad (B + D + \frac{5}{12})}\right] x^5 \dots \tag{50}$$

THE CONVERGENCE

To prove the convergence of the series (50), we arrange (48) in descending powers of λ,

$$\lambda^2 \left\{ a_\lambda \left(B + D + \frac{5}{12}\right) - a_{\lambda-2} (1 + D) + \tfrac{3}{4} a_{\lambda-4} - \tfrac{1}{6} a_{\lambda-6} \right\}$$

$$- \lambda . M + N = 0,$$

where M and N are independent of λ. (51)

Dividing by $\lambda^2 . a_{\lambda-6}$ and taking limit as $\lambda \to \infty$ and letting

$$\underset{\lambda \to \infty}{Lt} \frac{a_\lambda}{a_{\lambda-2}} = l, \tag{52}$$

relation (51) reduces to

$$f (l) \equiv l^3 \left(B + D + \frac{5}{12}\right) - l^2 (1 + D) + \tfrac{3}{4} l - \tfrac{1}{6} = 0. \tag{53}$$

By Descarte's rule of signs, it is observed that $f(l)$ has all the three roots positive.

Now we see that $f(l) < 0$ for $l = 0$ and $f(l) > 0$ when $l = 1$.

For all values of $l < 0$, $f(l)$ is always negative and for the values of $l > 1$, $f(l)$ remains positive. Therefore, the inferior and superior limits of l are 0 and 1 respectively. Thus the three roots of l lie between 0 and 1 and the convergence of the series (50) is established.

CONCLUSIONS

The magnetic field increases the frequency of pulsation. All the calculations made above are for the fundamental mode of vibrations. γ has been taken to be $\frac{5}{3}$, and the density ρ_c at the axis of the cylinder to be uniform throughout.

We find that ψ, the displacement function, decreases as $\frac{H_0^2}{H_s^2}$ increases. It also decreases with the increase of D as can be seen from the Tables 4 and 5 and the Graphs 3 and 5.

The relative change in the magnetic field $\frac{\delta H}{H}$ decreases with the increase in the frequeney of pulsation *i.e.* it decreases with the increase in the magnetic field. Also it sharply decreases as x increases, which implies that it is maximum at the axis and minimum on the surface.

Tables (1) and (2) give, respectively, the frequencies of pulsations for $\frac{H_0^2}{H_s^2} \leqslant 1$ and $\frac{H_0^2}{H_s^2} \geqslant 1$. Table (3) gives frequencies for both $\frac{H_0^2}{H_s^2} \lessgtr 1$. Tables (4) and (5) show the values of displacement function ψ for certain values of A. Table (6) gives the relative change $\left|\frac{\delta H}{H}\right|$ in magnetic field.

Graphs (1) and (2), respectively show the variation in frequencies of pulsations, with respect to $\frac{H_0^2}{H_s^2} < 1$ and $\frac{H_0^2}{H_s^2} > 1$. In graph (3), ψ for $A = 56{\cdot}80$, has been plotted against x and graph (4) shows the variation in the difference $\eta = \psi_2 - \psi_1$ for $A = 56{\cdot}80$ and $A = 54{\cdot}53$ with respect to x. Similarly in graphs (5) and (6), ψ for $\frac{H_0^2}{H_s^2} = 6$ and $\eta = \psi_2 - \psi_1$ for $\frac{H_0^2}{H_s^2} = 1$ and 6, respectively have been plotted against x. Graph (7) shows the variation in $\left|\frac{\delta H}{H}\right|$ against x.

TABLE 1 : $\frac{H_0^2}{H_s^2} \leqslant 1$

B	D	$\frac{H_0^2}{H_s^2}$	$A = \frac{1}{\gamma}\left[\frac{\sigma^2}{\pi \rho_c G} + 4\right]$
6	4·2	0·2	51·19
6	4·4	0·4	54·53
6	4·6	0·6	56·80
6	5·0	1·0	61·45

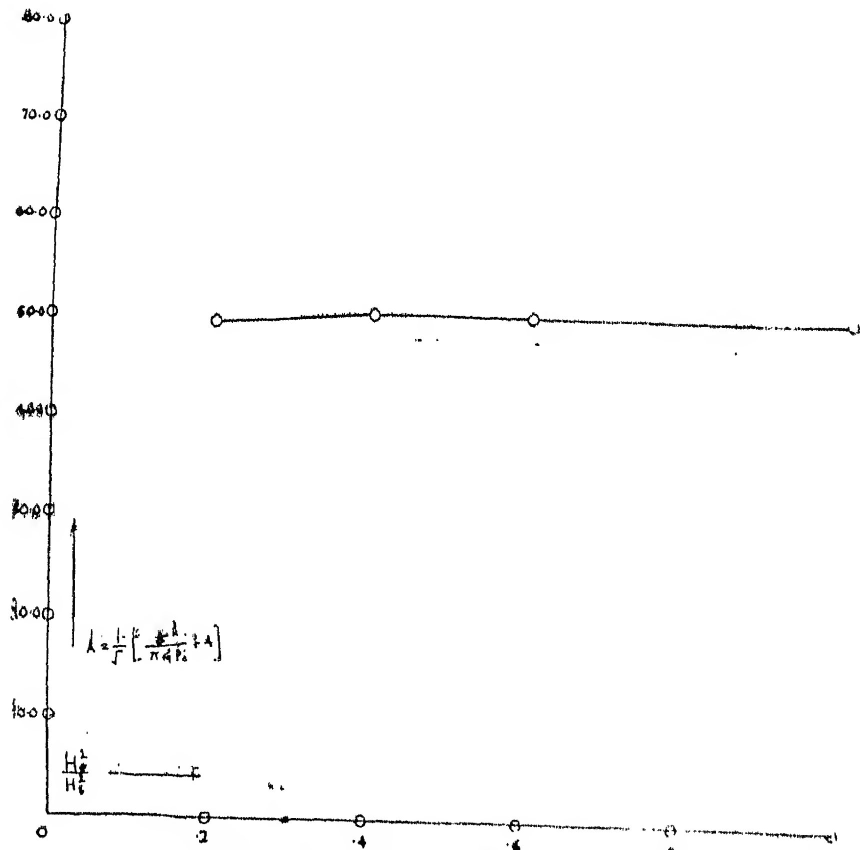


FIG. No 1 THE CURVE A AS A FUNCTION OF $\frac{H_0^2}{H_s^2}$

TABLE 2 : $\frac{H_0^2}{H_s^2} \geqslant 1$

B	D	$\frac{H_0^2}{H_s^2}$	$A = \frac{1}{\gamma}\left[\frac{\sigma^2}{\pi G \rho_c} + 4\right]$
6	5	1	61·245
6	6	2	72·335
6	8	4	94·430
6	10	6	116·048

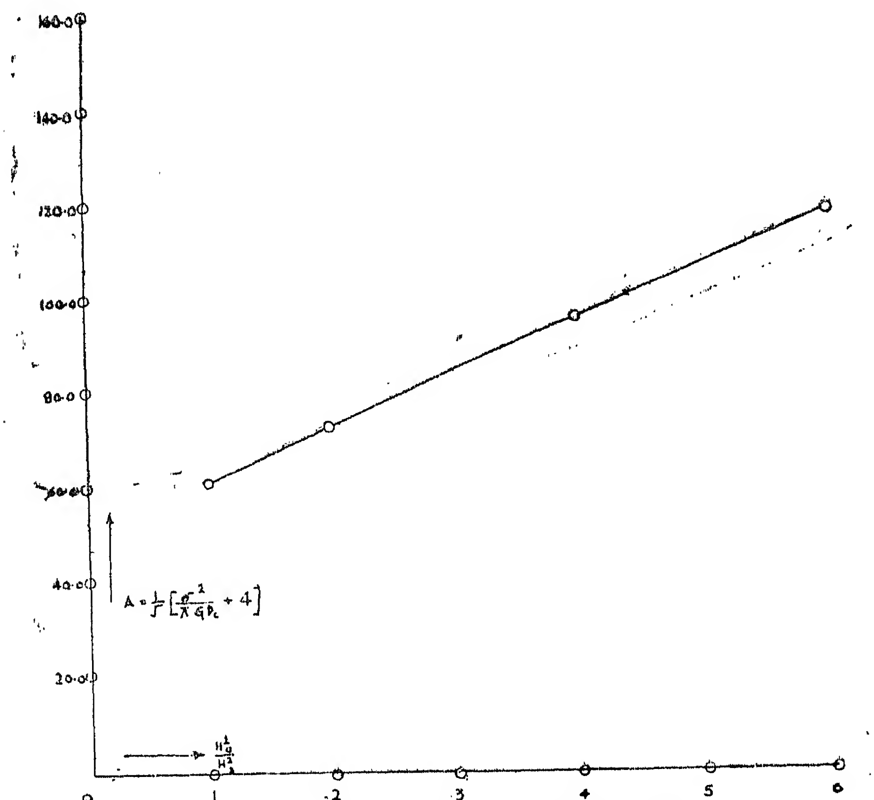


FIG. No 2 — THE CURVE A AS FUNCTION OF $\frac{H_0^2}{H_s^2}$

TABLE 3 : (Miscellaneous)

B	D	$\dfrac{H_0^2/6}{H_s}$	$A = \frac{1}{\text{[illegible]}}\left[\dfrac{\sigma^2}{\pi G\rho_c} + 4\right]$
3	4	4·0	49·001
5	4	0·8	49·820
5	10	8·0	113·980
10	10	2·0	118·095

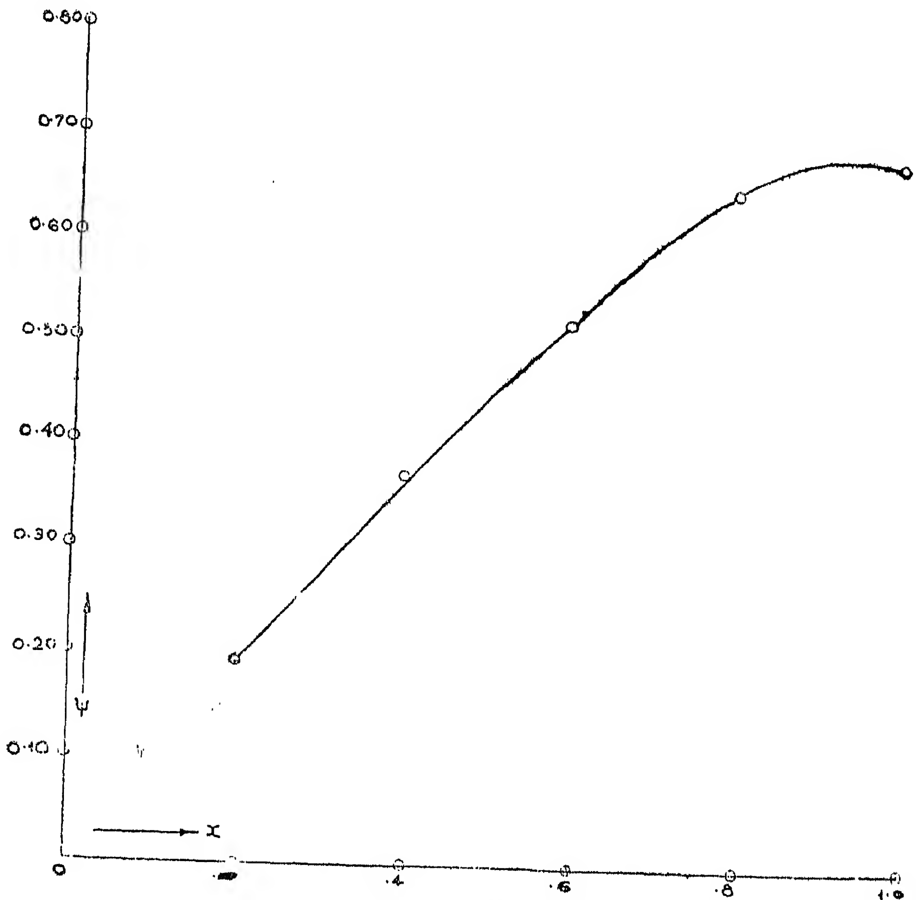


FIG. No 3 THE CURVE ψ AS A FUNCTION OF x FOR A = 56·80

TABLE 4

x	ψ_1 $B=6,\ D=4{\cdot}6$ $\frac{H_0^2}{H_s^2}=0{\cdot}6\ ;\ A=56{\cdot}80$	ψ_2 $B=6,\ D=4{\cdot}4$ $\frac{H_0^2}{H_s^2}=0{\cdot}4,\ A=54{\cdot}53$	$\psi_2-\psi_1=\eta$ The Difference
0·2	0·196924	0·197002	0·000078
0·4	0·376558	0·377149	0·000591
0·6	0·527634	0·529485	0·001851
0·8	0·652563	0·656448	0·003885
1·0	0·782230	0·788157	0·005927

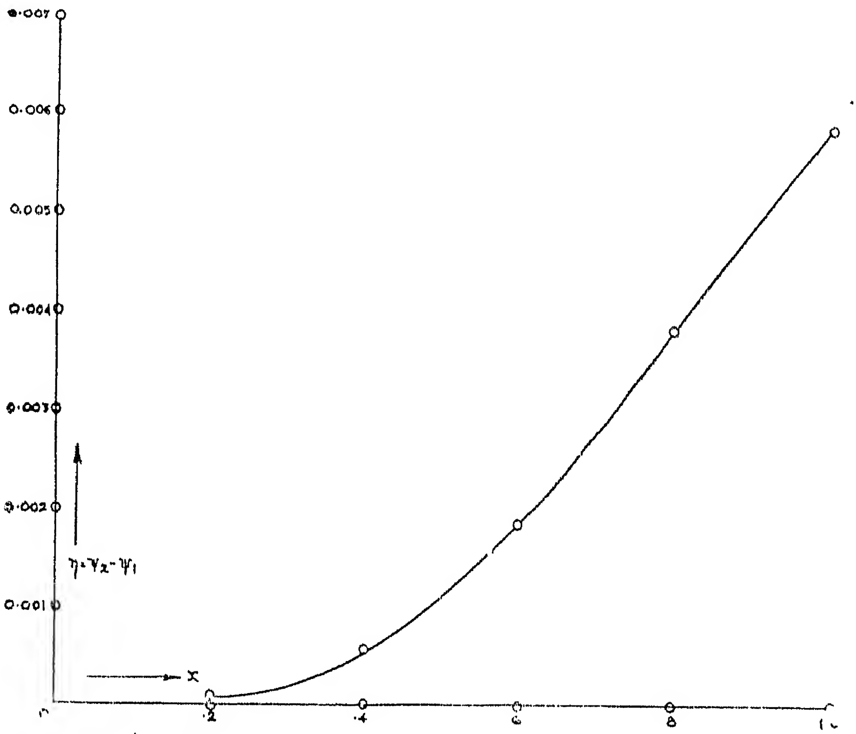


FIG. No 1 THE CURVE η AS A FUNCTION OF X FOR A = 56·00 AND A = 54·53

TABLE 5

x	ψ_1 $B = 6,\ D = 10$ $\dfrac{H_0^2}{H_s^2} = 6\ ;\ A = 116{\cdot}048$	ψ_2 $B = 6,\ D = 5$ $\dfrac{H_0^2}{H_s^2} = 1\ ;\ A = 61{\cdot}245$	$\eta = \psi_2 - \psi_1$ The Difference
0·2	0·195675	0·196790	0·001115
0·4	0·366934	0·375502	0·008568
0·6	0·497185	0·524331	0·027146
0·8	0·587206	0·645593	0·058387
1·0	0·680039	0·771613	0·091574

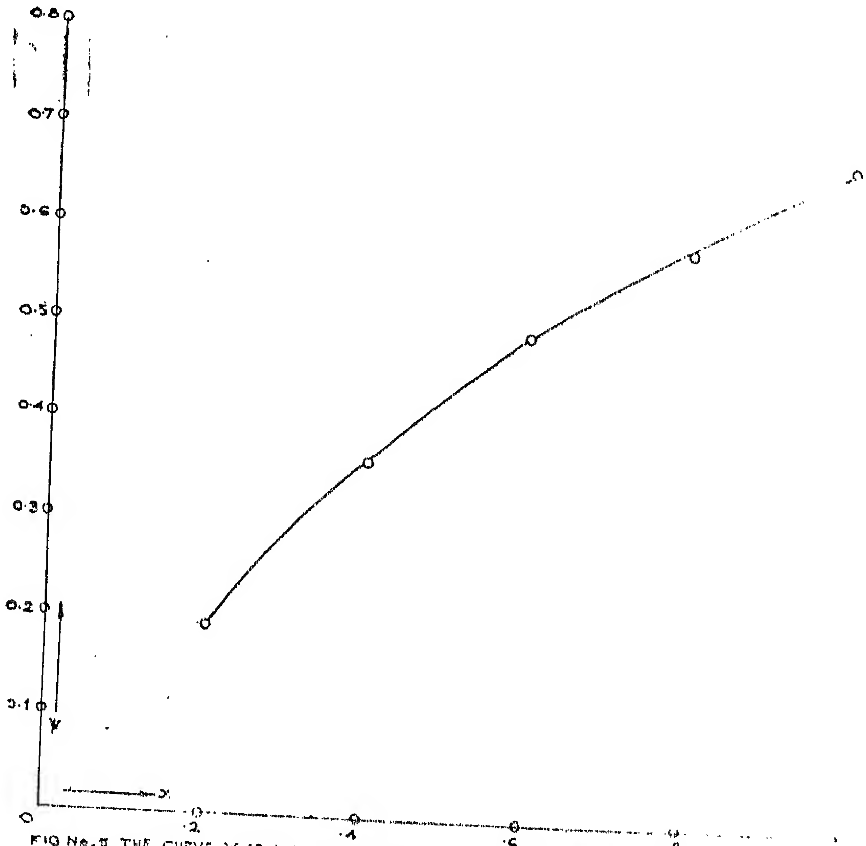


FIG No. 5 THE CURVE ψ AS A FUNCTION OF x FOR A = 116·048

TABLE 6

The relative change in H.

x	$\frac{\mid \delta H \mid}{H}$ for $A = 54{\cdot}530$	$\frac{\mid \delta H \mid}{H}$ for $A = 5{\cdot}6800$	$\frac{\mid \delta H \mid}{H}$ for $A = 61{\cdot}245$	$\frac{\mid \delta H \mid}{H}$ for $A = 116{\cdot}048$
0·2	2·280600	2·309000	1·906200	2·337400
0·4	1·854200	1·934700	1·649200	2·026700
0·6	1·343000	1·447500	1·294000	1·598050
0·8	0·918400	0·998000	0·939900	1·188900
1·0	0·788158	0·782225	0·771617	0·696586

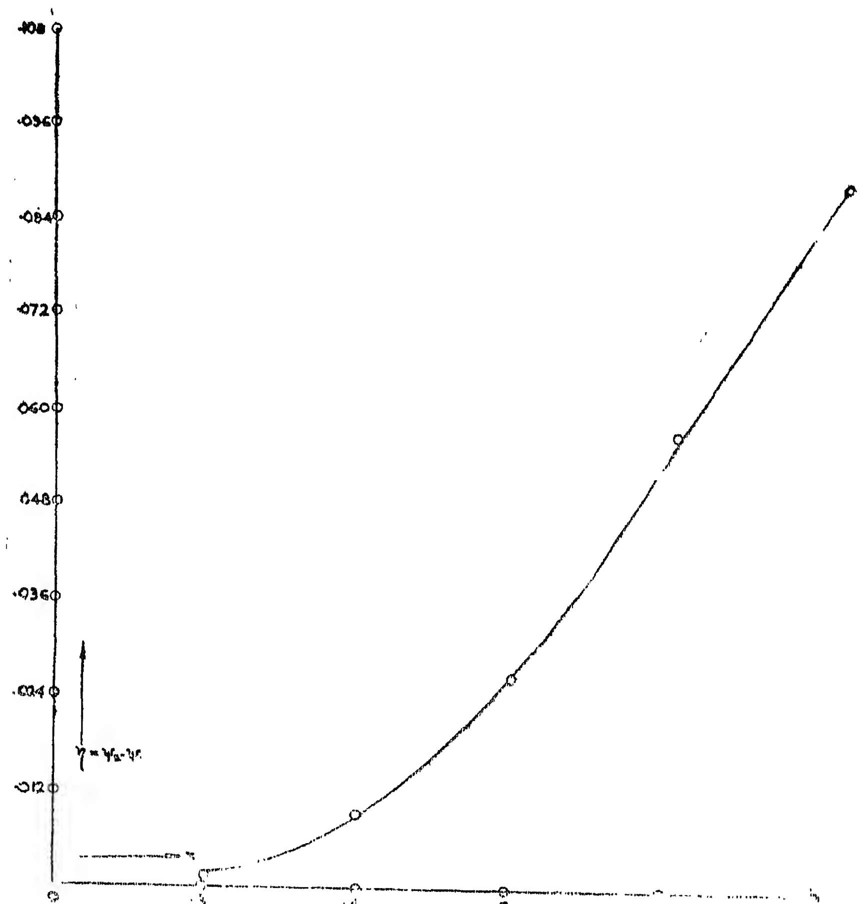


FIG. NO. 4 THE CURVE η AS A FUNCTION OF x FOR [illegible]

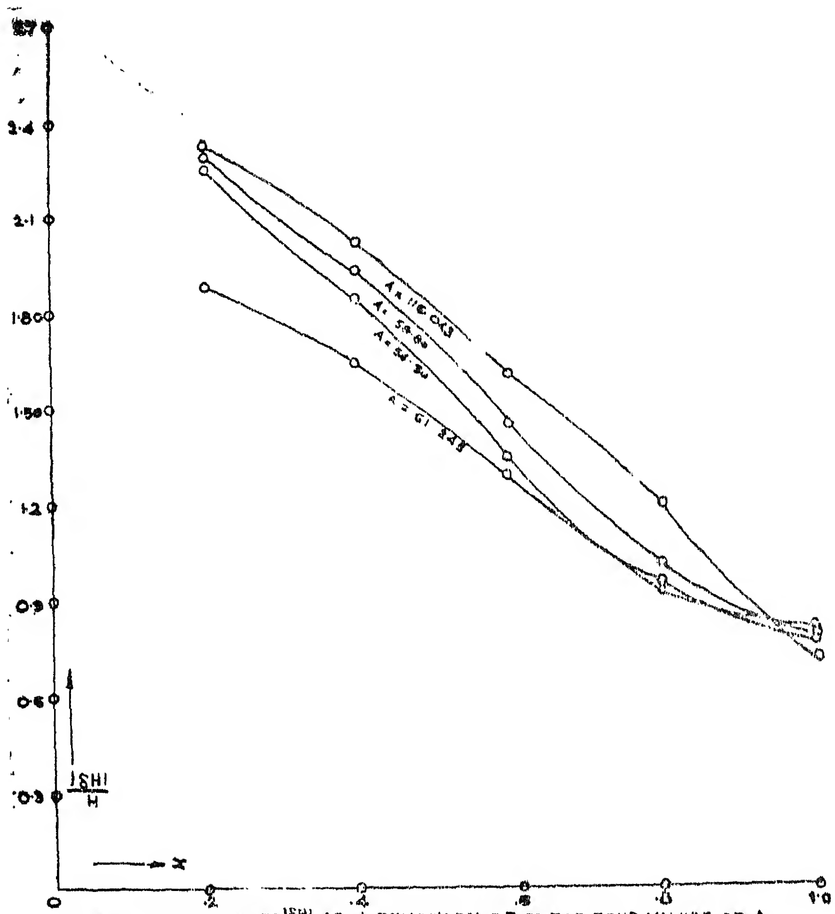


FIG. No 7 THE CURVES $\frac{|\delta H|}{H}$ AS A FUNCTION OF x FOR FOUR VALUES OF A

REFERENCES

1. Chandresekhar, S. and Fermi, E. *Astrophys. J.* 118, 116, (1953).
2. Lyttkens, E. *Astrophys, J.* 119, 413, (1954).
3. Chopra, K. P., and Talwar, S. P. *Proc. Nat. Inst. Sei.* 21 : 302 (1955).
4. Bhatnagar, P. L. and Nagpaul, S. P. *Z. Astrophy*, 43 : 273 (1957).
5. Chakraborty, B. B. and Rama Moorthy, P. *Z. Astrophy*, 49 : 186 (1960).
6. Tandon J. N. and Talwar, S. P. *Ind. J. Phys.* 32 : 317 (1958).
7. Raju P. K. and Talwar, H. S. *Z. Astrobhys.*, 52 : 31 (1961).
8. Gurm H. S. *Nat. Inst. India*, Sec. A, 27 : 319 (1961).

AN INTEGRAL INVOLVING PRODUCTS OF G-FUNCTIONS

By

R. K. SAXENA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 22nd August, 1964]

ABSTRACT

In this note an integral involving products of Meijer's G-functions is evaluated with the help of operational calculus, which is a generalization of SUNDARARAJAN'S result appeared recently in these proceedings (4, p. 98).

1. The formula to be proved is

$$\int_0^\infty t^{\sigma-1} \; G^{m,\,n}_{r,\,s}\left(at \,\middle|\, \begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right) \; G^{\alpha,\,\beta}_{\gamma,\,\delta}\left(b+t \,\middle|\, \begin{matrix} c_1,\dots,c_\gamma \\ d_1,\dots,d_\delta \end{matrix}\right) dt$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, b^r}{r\,!}$$

$$\times\, G^{\beta+m,\,\alpha+n+1}_{\delta+r+1,\,\gamma+s+1}\left(a \,\middle|\, \begin{matrix} 1-\sigma,\, a_1,\dots,a_n,\, 1+r-\sigma-d_1,\dots,\, 1+r-\sigma-d_\delta,\, a_{n+1},\dots,a_r \\ b_1,\dots,b_m,\, 1+r-\sigma-c_1,\dots,\, 1+r-\sigma-c_\gamma,\, 1+r-\sigma,\, b_{m+1},\dots,b_s \end{matrix}\right), \qquad (1)$$

where $R(\sigma+\min b_j) > 0$ for $j=1, 2, \dots, m$; $R(\sigma+\max a_i+\max c_k) < 2$ for $i=1, 2, \dots, n$; $k=1, 2, \dots, \beta$; $|\arg a| < (m+n-\frac{1}{2}r-\frac{1}{2}s)\,\pi$, $m+n>\frac{1}{2}r+\frac{1}{2}s$, $|\arg b| < \pi$ and $\alpha+\beta>\frac{1}{2}\gamma+\frac{1}{2}\delta$.

In what follows the conventional notation $\phi(p) \doteqdot h(t)$ will be used to denote the classical Laplace's integral

$$\phi(p) = p\int_0^\infty e^{-pt}\, h(t)\,dt. \qquad (2)$$

2. Proof of the formula.—Starting with (1, p. 222) we take

$$\phi_1(p) = p^{1-\sigma}\, G^{m,\,n+1}_{r+1,\,s}\left(\frac{a}{p} \,\middle|\, \begin{matrix} 1-\sigma,\, a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right)$$

$$\doteqdot t^{\sigma-1}\; G^{m,\,n}_{r,\,s}\left(at \,\middle|\, \begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right)$$

$$= h_1(t),$$

where $R(\sigma+\min b_j)>0$ for $j=1,2,\dots,m$, $|\arg a| < (m+n-\frac{1}{2}r-\frac{1}{2}s)\pi$, $m+n>\frac{1}{2}r+\frac{1}{2}s$, $R(p)>0$ and

$$\phi_2(p) = p\, G^{\alpha,\,\beta}_{\gamma,\,\delta}\left(p+b \,\middle|\, \begin{matrix} c_1,\dots,c_\gamma \\ d_1,\dots,d_\delta \end{matrix}\right)$$

$$\doteq \frac{1}{t} e^{-bt} \; G^{\alpha,\,\beta}_{\gamma+1,\,\delta} \left(t^{-1} \,\middle|\, \begin{matrix} c_1, \ldots, c_\gamma, 0 \\ d_1, \ldots, d_\delta \end{matrix} \right)$$

$$= h_2 (t),$$

where $R(p) > 0$, $R(b) > 0$, $R(a_j) < 1$ for $j = 1, 2, \ldots, \beta$, $\alpha + \beta > \frac{1}{2}(\gamma + \delta + 1)$.

Applying the Goldstein theorem of the operational calculus (3, p. 105) to the above operational pairs, we find that

$$\int_0^\infty t^{\sigma-1} G^{m,\,n}_{r,\,s} \left(at \,\middle|\, \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right) G^{\alpha,\,\beta}_{\gamma,\,\delta} \left(b + t \,\middle|\, \begin{matrix} c_1, \ldots, c_\gamma \\ d_1, \ldots, d_\delta \end{matrix} \right) dt$$

$$= \int_0^\infty t^{-\sigma-1} e^{-bt} G^{m,\,n+1}_{r+1,\,s} \left(\frac{a}{t} \,\middle|\, \begin{matrix} 1-\sigma, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right)$$

$$\times G^{\alpha,\,\beta}_{\gamma+1,\,\delta} \left(\frac{1}{t} \,\middle|\, \begin{matrix} c_1, \ldots, c_\gamma, 0 \\ d_1, \ldots, d_\delta \end{matrix} \right) dt.$$

Expand e^{-bt} in powers of b and interchangethe order of integration and summation, which is justifiable, the *r. h. s.* of the above relation then becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^r b^r}{r!} \int_0^\infty t^{\sigma-r-1} G^{m,n+1}_{r+1,s} \left(at \,\middle|\, \begin{matrix} 1-\sigma, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right)$$

$$\times G^{\alpha,\,\beta}_{\gamma+1,\,\delta} \left(t \,\middle|\, \begin{matrix} c_1, \ldots, c_\gamma, 0 \\ d_1, \ldots, d_\delta \end{matrix} \right) dt$$

If we now evaluate the above integral by Meijer's formula (2, p. 442)

$$\int_0^\infty t^{\sigma-1} G^{m,\,n}_{r,\,s} \left(t \,\middle|\, \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right) G^{\alpha,\,\beta}_{\gamma,\,\delta} \left(zt \,\middle|\, \begin{matrix} c_1, \ldots, c_\gamma \\ d_1, \ldots, d_\delta \end{matrix} \right) dt$$

$$= G^{\alpha+n,\,\beta+m}_{\gamma+r,\,\delta+s} \left(z \,\middle|\, \begin{matrix} c_1, \ldots, c_\beta, 1-b_1-\sigma, \ldots, 1-b_s-\sigma, c_{\beta+1}, \ldots, c_\gamma \\ d_1, \ldots, d_\alpha, 1-a_1-\sigma, \ldots, 1-a_r-\sigma, d_{\alpha+1}, \ldots, d_\delta \end{matrix} \right) \tag{3}$$

where $R(\sigma + \min b_i + \min d_j) > 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, \alpha$, $R(\sigma + \max a_k + \max c_l) < 2$ for $k = 1, 2, \ldots, n$, $l = 1, 2, \ldots, \beta$, $|\arg z| < (\alpha + \beta - \frac{1}{2}\gamma - \frac{1}{2}\delta)\pi$, $\alpha + \beta > \frac{1}{2}\gamma + \frac{1}{2}\delta$, $m + n > \frac{1}{2}r + \frac{1}{2}s$, we arrive at the result.

On taking $\alpha = \delta = 1$, $\beta = \gamma = 2$, $c_1 = 1 - k$, $c_2 = 1 - m$ and using the formula

$$G^{2,\,1}_{1,\,2} \left(x \,\middle|\, \begin{matrix} 1 \\ k,\ m \end{matrix} \right) = E(k, m :: x), \tag{4}$$

we obtain a result given by Sundararajan (4, p. 98).

REFERENCES

1. Erdelyi, A. *et al.* Tables of integral transforms, **1** : (1954).
2. Erdelyi, A. *et al.* Tables of integral transforms, **2** : (1954).
3. Goldstein, S. *Proc. London Math. Soc.*, **34** : 103–125, (1932).
4. Sundararajan, P. K. *Proc. Nat. Acad. Sci.*, **34** ; 97–104, (1964).

ON SOME UNIQUENESS THEOREMS FOR ORDINARY NON-LINEAR DIFFERENTIAL EQUATIONS

By

M. RAMA MOHANA RAO

Department of Mathematics, College of Engineering, Osmania University, Hyderabad

[Received on 23rd February, 1965]

ABSTRACT

There is an increased interest in uniqueness theorems for the first as well as the second order differential equations. Kearns has attempted some uniqueness problems by using a lemma known as Bellman's lemma. We shall attempt some uniqueness problems for the solutions of the non-linear differential equations of second order of the type

$$y'' + \phi\,[t, y\,(t)\,] = f\,(t)$$

which are generalisations of Kearn's results. In § 4 a result for uniqueness of the solutions of a non-linear perturbed differential equation of the type

$$y' = A\,(t)\,y + f\,(t, y)$$

is obtained.

§ **1.** There is an increased interest in uniqueness theorems for first as well as second order differential equations. Kearns[3] has attempted some uniqueness problems by using a lemma known as Bellman's lemma *cf.*[1] We shall attempt some uniqueness problems for the solutions of the non-linear differential equations of second order which are generalisations of.[3] In § 4 a result for uuiqueness of the solution of a non-linear perturbed differential equation is obtained. We will assume once for all the conditions necessary for the existence of the integral which appear in the calculations.

Lemma 1. If $u(t)$ and $v(t)$ are non-negative, if k is any positive quantity and if

$$u(t) \leqslant k + \int_0^t u(s)\; v(s)\; ds$$

then
$$u(t) \leqslant k\; exp\left(\int_0^t v(s)\; ds\right)$$

This lemma is due to Bellman *cf.*[1]

§ **2.** Consider the differential equation

(2·1) $\quad y''\,(t) + \phi\,(t, y\,(t)\,) = 0$

where ϕ is a continuous function in both the variable on $t \geqslant 0, y \geqslant 0$.

We will assume a continuous solution of (2·1) satisfying the initial conditions $y\,(0) = y_0, y'\,(0) = y_1$.

Now

$$\begin{aligned} y\,(t) - ty_1 - y_0 &= \int_0^t \left(\int_0^u y''(s)\; ds\right) du \\ &= -\int_0^t \left[\int_0^u \phi\,(s, y\,(s)\,)\; ds\right] du \end{aligned}$$

Integrate the last iterated integral by parts, we obtain

$$\int_0^t \left(\int_0^u \phi(s, y(s))\, ds \right) du = t \int_0^t \phi(s, y(s))\, ds - \int_0^t u\, \phi(u, y(u))\, du$$

changing the variable of integrations from s to u in the first integral on the right hand side of this equation, we may write

$$\int_0^t \left[\int_0^u \phi(s, y(s))\, ds \right] du = \int_0^t (t-u)\, \phi(u, y(u))\, du$$

Therefore equation (2·1) is equivalent to

$$y(t) = y_0 + t y_1 - \int_0^t (t-s)\, \phi(s, y(s))\, ds$$

Suppose $\quad | \phi(t, y(t)) | \leqslant k\, | y(t) |$ where k is a positive constant

Hence for $\quad t \geqslant 0$

$$| y(t) | \leqslant | y_0 | + t\, | y_1 | + \int_0^t (t-s)\, k\, | y(s) |\, ds$$

If $\quad 0 \leqslant t \leqslant 1$

$$| y(t) | \leqslant | y_0 | + | y_1 | + k \int_0^t (t-s)\, | y(s) |\, ds$$

and therefore by applying lemma 1, we have

$$(2{\cdot}2a) \qquad | y(t) | \leqslant (| y_0 | + | y_1 |)\, exp \left(\int_0^t k\, (t-s)\, ds \right)$$

on the other hand if $t > 1$

$$| y(t) | \leqslant t\, (| y_0 | + | y_1 |) + t \int_0^t k\, | y(s) |\, ds$$

$$i.e. \quad | y(t) | / t \leqslant (| y_0 | + | y_1 |) + \int_0^t k\, | y(s) |\, ds$$

By applying lemma 1, we have

$$(2{\cdot}2b) \qquad \frac{| y(t) |}{t} \leqslant (| y_0 | + | y_1 |)\, exp \left(\int_0^t ks\, ds \right)$$

$$\leqslant (| y_0 | + | y_1 |)\, exp \left(\frac{kt^2}{2} \right)$$

Now we are in a position to prove the following theorem.

Theorem 1. Let $g(t, u(t))$ be a continuous function in both variables for $t \geqslant 0$, $u \geqslant 0$ and if for $t \geqslant 0$, there exists a continuous solution of the equation

$$(2{\cdot}3) \qquad u''(t) + g(t, u(t)) = f(t)$$

satisfying $u(0) = k_0$, $u'(0) = k_1, (k_0, k_1 > 0)$ and further

satisfy $|\phi(t, u_1 - u_2)| \leqslant k|u_1 - u_2|$

where $g(t, u_1) - g(t, u_2) = \phi(t, u_1 - u_2)$

and k is a constant, u_1, u_2 are different functions of t, then that solution is unique.

Proof. Suppose $u_1(t)$ and $u_2(t)$ are two solutions of (2·3).

Then $y(t) = u_1(t) - u_2(t)$ satisfy (2·1) with initial conditions $y(0) = (0), y'(0) = 0$.

By (2·2a) or (2·2b) $|y(t)| \leqslant 0$

which implies $y(t) \equiv 0$.

Hence $u_1(t) = u_2(t)$.

§ **3.** It is also clear that the above argument will apply if the Lipschitz constant k is replaced by a function $k(t)$ which is obviously a generalisations of Nagumo condition *cf.*[4] for uniqueness criterion. Then the above theorem runs as follows.

Theorem 2. Let $g(t, u)$ be a continuous function in both the variables for $t \geqslant 0, u \geqslant 0$ and if for $t \geqslant 0$ there is a continuous solution of the equation

$$u'' + g(t, u) = f(t)$$

satisfying $u(0) = k_0, u'(0) = k_1$ and further

satisfy $|\phi(t, u_1 - u_2)| \leqslant k(t)|u_1 - u_2|$

where $g(t, u_1) - g(t, u_2) = \phi(t, u_1 - u_2)$

and $k(t) > 0$, u_1 and u_2 are two different functions of t, then that solution is unique.

§ **4.** In this section, we shall consider the non-linear perturbed differential system

(4·1) $u'(t) = A(t)u + g(t, u(t))$

where $A(t)$ is $n \times n$ matrix, continuous for $t \geqslant 0$,

and $g(t, u)$ is a continuous vector function for $t \geqslant 0$, $|u| \leqslant c$.

We first prove a lemma which is slight modification of a standard type of lemma *cf.*[2]

Lemma 2. In the differential equation (4·1), let

(4·2) $$k(t) = |A(t)|$$

satisfy

(4·3) $$P(t) = \int_0^t k(s)\, ds$$

let $\phi(t, r)$ be a continuous scalar function for $t \geqslant 0$, $r \geqslant 0$

and satisfy

(4·4) $$|g(t, u)| \leqslant \phi(t, |u|)|u|$$

and

(4·5) $$d_1 = \int_0^t \phi(s, |u|)ds$$

Then $\quad |u(t)| \leqslant |u(0)|\, exp\,(d_1 + p(t))$ for $t \geqslant 0$

Proof. Let $u(t)$ be a solution of (4·1)

$$|u'(t)| \leqslant |A(t)|\;|u| + |g(t, u)|$$

Therefore from (4·2) and (4·4)

$$|u|' \leqslant k(t)|u| + \phi(t, |u|)|u|$$

Let

(4·6) $$v(t) = \frac{|u|}{exp\,(p(t))}$$

$$v'(t) = \frac{|u|'}{exp\,(p(t))} - \frac{|u|\,p'(t)}{exp\,[p(t)]}$$

$$= \frac{|u|'}{|u|}\, v(t) - p'(t).\, v(t)$$

i.e. $$\frac{v'(t)}{v(t)} = \frac{|u|'}{|u|} - p'(t) \leqslant \phi(t, |u|)$$

Integrating between 0 to t, we get

$$v(t) \leqslant v(0)\, exp\left(\int_0^t \phi(s, |u|)\,ds\right)$$

$$\leqslant v(0)\, exp\,[d_1] \text{ from (4·5)}$$

But from (4·3) and (4·6)

$$p(0) = 0, \qquad v(0) = |u(0)|$$

Therefore

(4·7) $$|u(t)| \leqslant |u(0)|\, exp\,(d_1 + p(t))$$

We are now in a pasition to prove the following theorem.

Theorem 3. Let (*i*) $A(t)$ be an $n \times n$ matrix, continuous for $t \geqslant 0$.

(*ii*) $f(t, x)$ be a continuous vector function for $t \geqslant 0$, $|x| \leqslant r$

If there is a continuous solution of the equation

(4·8) $$x' = A(t)x + f(t, x), \qquad x(0) = x_0$$

and satisfy

$$|g(t, x_1 - x_2)| \leqslant \phi(t, |x_1 - x_2|)\,|x_1 - x_2|$$

where $\quad f(t, x_1) - f(t, x_2) = g(t, x_1 - x_2)$

where x_1 and x_2 are two different function of t. Then, that solution is unique

Proof. Suppose x_1 (:) and x_2 (t) are two solutions of (4·8).

Then $u(t) = x_1(t) - x_2(t)$ satisfies (4·1) with initial condition $u(0) = 0$.

By (4·7) $\quad |u(t)| \leqslant 0$ which implies $u(t) = 0$.

Therefore $\quad x_1(t) = x_2(t)$.

REFERENCES

1. Bellman, R. Stability Theory of ordinary differential equations. *McGraw Hill Company, Inc*, 35, (1953).
2. Hartman, P. and Olech, C. On Global asymptotic stability of solutions of differential equations. *Tran. A. M. S.*, Vol. 104, 154–178, (1962).
3. Kearns, D. A. Elementary uniqueness for ordinary differential equations. *Amer. Monthly*, Vol. 68, 275–277, (1961).
4. Nagumo. *Jap. Jr. Math.*, 3 : 107–112, (1926).

STUDIES ON HYDROUS BERYLLIUM OXIDE

ELECTROMETRIC AND CONDUCTOMETRIC STUDIES ON THE PRECIPITATION OF HYDROUS BERYLLIUM OXIDE

By

RUP DUTTA *and* SATYESHWER GHOSH*

Chemical Laboratories, University of Allahabad, Allahabad

[Received on 23rd February, 1965]

ABSTRACT

Various mixtures of beryllium sulphate and different amount of alkali have been prepared, ratio of Be (II) : OH^- varied from 1: 0·0 to 1: 2·6, and the pH and conductivity values determined. In the pH curves two inflexions occur, one at Be (II) : OH = 1:1 and the other at Be (II) : OH^- = 1 : 1·92. The first inflexion represents a stage when precipitate of beryllium oxide appears and the other, when the precipitation is complete. The conductivity curves are similar though the first inflexion is not very distinct.

Ageing has little or no effect on the hydrous oxide. However, on prolonged ageing of the mixtutes 1 : 2 (Be (II) : OH^-) shows again a decrease in the pH and conductivity values. This indicates the formation of beryllates by the removal of alkali from the solution.

In earlier communication[1] the studies on the precipitation of hydrous beryllium oxide have been reported. The pH and conductometric measurements during the precipitation of hydrous oxides enable us to obtain valuable informations. Britton[2] has carried out an extensive work on the precipitation of hydrous oxides by pH measurements. He has shown that the precipitation of hydrous oxides commences when a definite pH value, within narrow limits, has been attained in the system.

In this paper, results obtained from the electrometric and conductometric studies on the precipitation of hydrous beryllium oxide from a beryllium sulphate solution with the addition of alkali have been reported. Incidentally, the effect of ageing *i.e.* changes in conductance and pH on allowing the systems to stand for different intervals of time, has also been noted.

EXPERIMENTAL

Standard solutions of B. D. H., A. R. beryllium sulphate and E. Merck sodium hydroxide were prepared. Various samples of beryllium hydroxide were obtained by adding varying amount of sodium hydroxide to a known quantity of beryllium sulphate solution. After vigourously shaking each of the contents, the pH of the supernatant liquids was determined after one hour using L and N direct reading pH meter, and then conductivity of the same solutions were determined. The mixtures of beryllium sulphate and sodium hydroxide were allowed to age for (1) 24 hours, (2) 168 hours and (3) 720 hours and their pH and conductivity were noted again. The results are presented in the following table :

Present Address : Department of Chemistry, University of Jabalpur, Jabalpur.

TABLE 1

Changes in pH and Electrical Conductance with ageing of the mixtures

Beryllium Sulphate 0·1M = 30ml

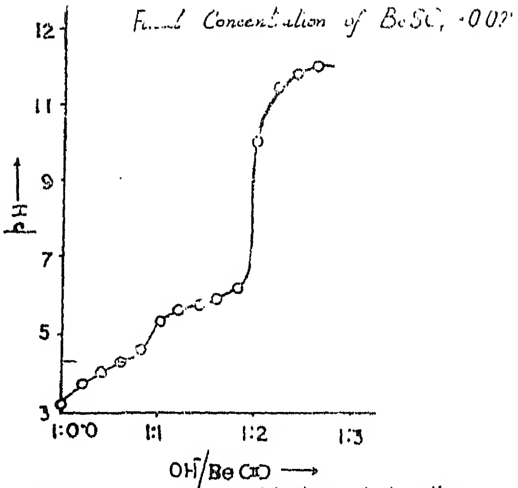
Total Volume = 100ml

Ratio Be (II):OH^-	Ageing of the Mixtures for: 1 hour pH	1 hour Sp. Cond. X10^4mhos	24 hours pH	24 hours Sp. Cond. X10^4mhos	168 hours pH	168 hours Sp. Cond. X10^4mhos	720 hours pH	720 hours Sp. Cond. X10^4mhos
1 : 0·0	3·20	50·50	3·20	50·50	3·20	50·50	3·15	50·20
1 : 0·2	3·75	50·30	3·75	50·30	3·65	50·30	3·70	50·30
1 : 0·4	4·00	52·27	3·95	52·15	3·95	52·15	3·95	52·35
1 : 0·6	4·30	55·83	4·200	52·80	4·20	55·83	4·20	55·83
1 : 0·8	4·60	59·05	4·55	59·05	4·50	59·05	4·50	50·05
1 : 1·0	5·35	62·70	5·30	62·82	5·25	63·10	5·25	63·20
1 : 1·2	5·70	66·96	5·65	66·95	5·65	66·96	5·60	67·00
1 : 1·4	5·75	72·30	5·70	72·48	5·65	72·60	5·65	72·60
1 : 1·6	5·90	76·74	5·80	77·10	5·80	77·20	5·80	77·20
1 : 1·8	6·20	82·86	6·20	82·20	6·15	82·25	6·10	82·25
1 : 2·0	10·05	89·16	10·05	89·18	10·00	89·16	7·60	89 28
1 : 2·2	11·40	104·34	11·45	110·46	11·50	110·46	11·40	103·00
1 : 2·4	11·75	122·22	11·80	122·20	11·80	123·18	11·70	118·03
1 : 2·6	11·90	140 00	11·95	140·20	11·95	141·66	11·85	128·00

CONCLUSION

The precipitation of hydrous beryllium oxide and the effect of ageing on the system have been followed by pH and conductometric measurements. The result show that the precipitation of hydrous beryllium oxide commences just after one equivalent of sodium or potassium or ammonium hydroxide has been added and the complete precipitation occurs at Be (II) : OH^- = 1 : 1·92 *i.e.* with less than the theoretical value of alkali. It is thus in conformity with the earlier work of Britton. It is found in the pH curves (Fig. 1) that two inflexions occur one at Be (II) : OH^- = 1 : 1 and other at Be (II) : OH^- = 1 : 1·92. The first inflexion represents a stage when precipitate of beryllium oxide appears and the other, when the precipitation is complete. Similar breaks occur in the conductance curves also though the first inflexion is not very sharp.

It will be seen from the Table 1 that ageing has little or no effect on the hydrous beryllium oxide. It is interesting to note that the pH after 720 hours of ageing, where Be (II) : OH^- is 1 : 2, falls sharply from 10·05 to 7·6 which is probably due to the maximum liberation of acid from the hydrous oxide.



Variation in pH on titration of beryllium sulphate solution with alkali.

On prolonged ageing of the mixtures beyond the ratio 1 : 2 (Be (II) : OH^-) show a decrease in the pH and the conductivity values is observed. This indicates the formation of beryllates, by the reaction between alkali and beryllium oxide, as a result of which the alkali diminishes.

ACKNOWLEDGEMENT

One of the authors (R. D.) is grateful to Council of Scientific and Industrial Research for the award of a fellowship.

REFERENCES

1. Dutta, R. and Ghosh, S. Communicated.
2. Britton, H. T. S. Annual Report, *Chem. Soc.* (*London*) 40, 44, (1943).

WAVES IN A HEAVY INCOMPRESSIBLE FLUID OF FINITE DEPTH AND OF VARIABLE DENSITY II

By

P. D. ARIEL *and* P. K. BHATIA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 23rd March, 1965]

§1. Abstract

We have studied here the problem of waves in a stratified viscous incompressible fluid in the presence of a uniform magnetic field $\vec{H}_0$ in the direction of z-axis *i.e.* in a direction parallel to the force of gravity. The fluid is assumed to be of finite depth and to be confined between two surfaces which are both rigid and ideal conductors. The buoyancy forces are not taken into consideration. An explicit solution has been obtained.

§2. Introduction

Hide (1955) studied the problem of waves in a heavy, incompressible, viscous, electrically conducting fluid in the presence of a uniform magnetic field $\vec{H}_0$ directed in a direction paralled to the force of gravity. He considered the case when the fluid is of finite depth d and is stratified in the verticle according to the law

$$\rho_0 = \rho_1 \exp [\beta z] \quad \ldots. (1)$$

where ρ_1 and β are constants. He also assumed that the fluid is confined between two surfaces which are both free. He considered the following three cases :

(1) Waves in the absence of magnetic field, (Hide, 1955*a*)

(2) Waves in the absence of buoyancy forces, (Hide, 1955)

(3) Waves in an ideal conductor in the presence of buoyancy forces, (Hide, 1955)

In all the three cases the bounding surfaces were both taken to be free.

In the present work we have studied the similar problem of waves in a heavy viscous incompressible fluid of finite depth d and stratified according to the law (1), confined between two surfaces which are both rigid and ideal conductors. We have considered the case when a uniform magnetic field $\vec{H}_0$ in the direction of z-axis is present and buoyancy forces are absent. An explicit solution has been obtained for this case.

The two cases corresponding to cases (1) and (3), considered by Hide, when the bounding surfaces are both rigid and ideal conductors have been discussed earlier in a separate paper (Paper I) by the authors.

3. Formulation of the problem

Proceeding as in Paper I we obtain [Equation (26) Paper 1]

$$y^2 + \frac{2\,(16 + 8x^2 + 3x^4)}{4 + 3x^2}\,y - \frac{12Gx^2}{4 + 3x^2} + \frac{4Qy}{y + 2P\,(4 + x^2)}\,\frac{4 + x^2}{4 + 3x^2} = 0, \quad \ldots\ldots \quad (2)$$

where

$$G \equiv \frac{g\beta d^4}{\pi^4 \nu^2 s^2}, \quad \ldots\ldots \quad (3)$$

$$Q \equiv \frac{kH_0^2 d^2}{\pi^3 \rho_1 \nu^2 s^2}, \quad \ldots\ldots \quad (4)$$

and

$$P \equiv \frac{\eta}{\nu} \quad \ldots\ldots \quad (5)$$

In equation (2) y denotes the dimensionless growth rate and x denotes the dimensionless wave number and they are given by

$$x = \frac{kd}{\pi s}, \; y = n\,\frac{2d^2}{\nu\pi^2 s^2} \quad \ldots\ldots \quad (6)$$

Here G has a form of Grashoff number which to some extent determines the relative importance of buoyancy forces and viscous forces, P measures the relative amounts of damping due to the electrical resistance and viscosity while Q is a suitable measure of magnetic field strength.

In deriving equation (2) we assumed that

$$|\,\beta d\,| << 1. \quad \ldots \quad (7)$$

Also in order that the boundary conditions of the problem

$$w = Dw = 0\,,\; D = d/dz, \quad \ldots\ldots \quad (8)$$

and

$$h = 0 \quad \ldots\ldots \quad (9)$$

are satisfied we assumed the trial functions for $w(z)$ and $h(z)$ as

$$w(z) = A\,(1 - \cos lz), \quad \ldots\ldots \quad (10)$$

$$h(z) = B \sin lz \quad \ldots\ldots \quad (11)$$

where A and B are constants and l is given by, from boundary conditions (8) and (9),

$$l = 2\pi s/d \quad \ldots\ldots \quad (12)$$

s being an integer.

In equations (10) and (11) $w(z)$ and $h(z)$ denote respectively the z-component of velocity vector and z-component of the perturbation in the magnetic field.

Equation (2) is a cubic in y, but can be reduced to quadratic with facility in the following three cases (I) $Q = 0$ or $P \to \infty$ (II) $P = 0$ and (III) $G = 0$.

The cases (I) and (II) have already been discussed by the authors in Paper (I). Presently we consider the case (III).

§ 4. Waves in the absence of Buoyancy forces

This is the case when $G = 0$, For this case equation (2) becomes

$$y^2 + \frac{2(16 + 8x^2 + 3x^4)}{4 + 3x^2} y + \frac{4Qy}{y + 2P(4 + x^2)} \cdot \frac{4 + x^2}{4 + 3x^2} = 0, \qquad \ldots . (13)$$

and if $y \neq 0$,

$$y^2 + 2\left\{ P(4 + x^2) + \frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right\} y + \frac{4(4 + x^2)}{4 + 3x^2} \{ Q + P(16 + 8x^2 + 3x^4) \} = 0. \qquad \ldots . (14)$$

The solutions of equation (14) are given by

$$y = -\left\{ P(4 + x^2) + \frac{16 + 8x^2\ 3x^4}{4 + 3x^2} \right\} \pm \left[\left\{ P(4 + x^2) - \frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right\}^2 - \frac{4Q(4 + x^2)}{4 + 3x^2} \right]^{\frac{1}{2}} \qquad \ldots . (15)$$

Since both P and Q are positive, therefore y never has a positive real part. Hence the equilibrium is stable. Whether it is restored periodically or aperiodically depends on the sign of the quantity under the radical sign.

$$\text{If } \phi = \phi(x) = \frac{4 + 3x^2}{4(4 + 3x^2)} \left\{ P(4 + x^2) - \frac{16 + 8x^2 + 3x^4}{4 + 3x^2} \right\}^2, \qquad \ldots . (16)$$

then it follows, from equation (15), that the motion is periodically or aperiodically damped according as

$$Q \gtrless \phi \qquad \ldots . (17)$$

In Graph 1, $\phi(x)$ is plotted as a function of x for the values of Plying in the range $\sqrt{3} - 1 \leqslant P \leqslant 1$.

Graph 2 gives the variation of ϕ against x for those values of P which lie outside the range $\sqrt{3} - 1 < P < 1$. In particular the graph is drawn for $P = 0{\cdot}5$, $\sqrt{3} - 1$, 1, 1·5.

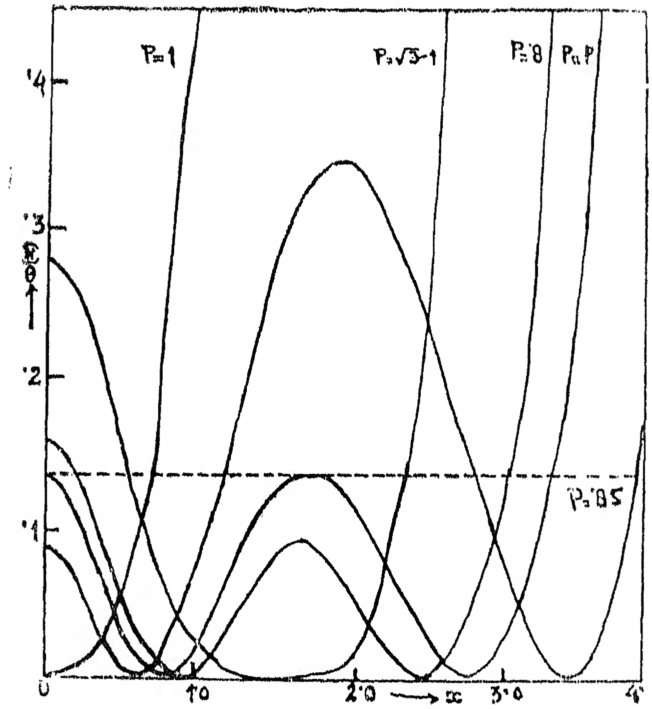
According to the criterion (17) we have oscillations only for those values of x for which a line drawn parallel to x-axis at a distance Q above it lies above the curve ϕ. For all other values of x the equilibrium is restored aperiodically.

Section 1. Investigations for the range $\sqrt{3} - 1 < P < 1$.

(a) *Characteristics of the Graph* 1 :

From the graph 1 we see that when P lies in the range $\sqrt{3} - 1 < P < 1$ $\phi(x)$ falls from its value $4(P - 1)^2$ at $x = 0$ to $\phi = 0$ at $x = x'$. The curve rises from this point till it reaches ϕ_E determined by

$$\phi_E = \frac{\{ 16(P - 1) + 8(2P - 1) x_E^2 + 3(P - 1) x_E^4 \}^2}{4(4 + x_E^2)(4 + 3x_E^2)}, \qquad \ldots . (18)$$



Graph 1. Plot of $\phi(x)$ against x for $\sqrt{3}-1 \leqslant P \leqslant 1$.

at $x = x_E$ given by

$$9x_E^6(P-1) + 72x_E^4(P-1) + 16x_E^2(11P-7) + 128P = 0. \qquad \ldots. (19)$$

The curve then again drops monotonically from ϕ_E to zero at $x = x''$ where x' and x'' are the positive roots of the equation

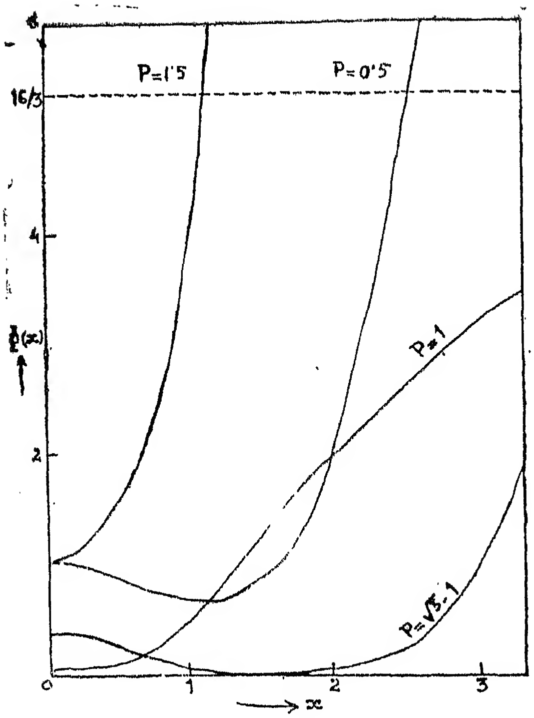
$$P(4+x^2) - \frac{16+8x^2+3x^4}{4+3x^2} = 0. \qquad \ldots. (20)$$

The curve now again rises and tends to infinity as x assumes larger values.

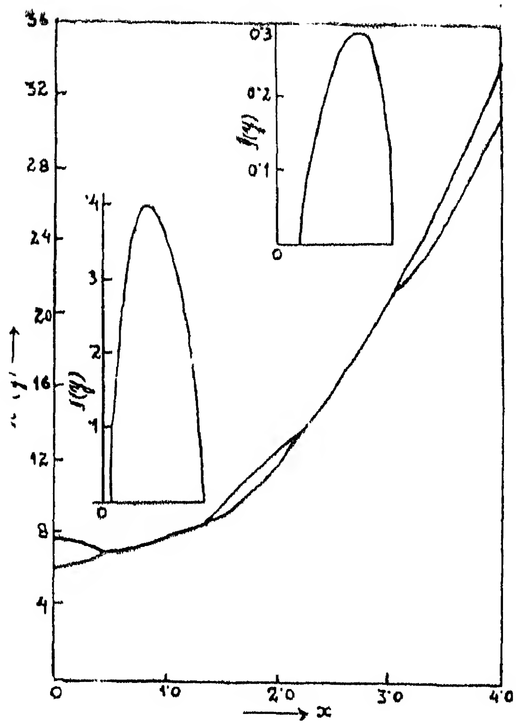
From the graph it is seen that we are to distinguish between two cases (*A*) $P > P^*$ and (*B*) $P < P^*$ where P^* is that value of P for which $\phi_E = \phi_{x=0}$. This implies that $P = P^*$ the equation $\phi_E = \phi_{x=0}$ should have two equal roots.

Thus $$9p^4 + 12p^3 - 33p^2 - 72p - 37 = 0, \qquad \ldots. (21)$$

where $$p = \frac{2(2P-1)}{3(1-P)} \qquad \ldots. (22)$$



Graph 2. Plot of ϕ (x) against x for $P \leqslant \sqrt{3}-1$ and $P \geqslant 1$



Graph 3. The growth rate y as a function of x for P = 0·8, Q = 0·05

Equation (21) is a biquadratic in P, therefore gives four values of P. However we choose only that value of P which lies inside the range $\sqrt{3}-1 < P < 1$. Thus P^* = ·8138.

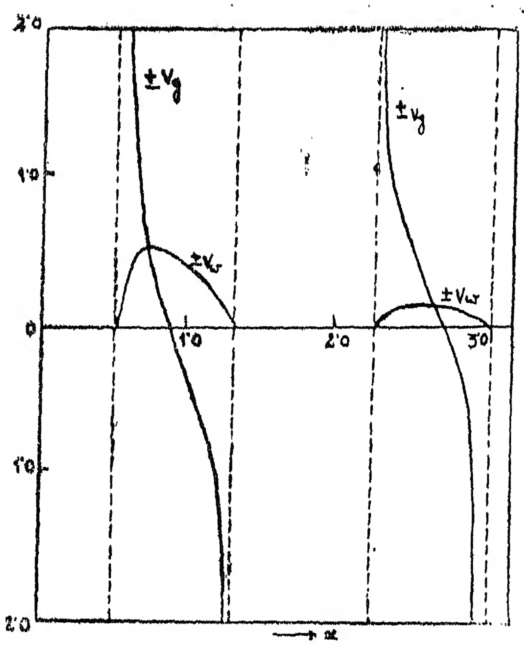
(*b*) *Conclusion*—

According to criterion (17) we conclude that for case (*A*) $P > P^*$

(*i*) When $0 < Q < 4\,(P-1)^2$, there are four values of x say x_1, x_2, x_3, x_4 for which $Q = \phi$. The oscillations arise for the ranges $x_1 < x < x_2$ and $x_3 < x < x_4$. The motion is aperiodic elsewhere.

(*ii*) When $4\,(P-1)^2 < Q < \phi_E$. In this range of Q there are three values of x denoted by x_5, x_6, x_7 for which $Q = \phi$. The oscillations therefore, arise for the ranges $0 < x < x_5$ and $x_6 < x < x_7$. The motion is aperiodic elsewhere.

(*iii*) Where $Q > \phi_E$. For this range of Q, there is only one value of x (say x_8) for which $Q = \phi$. The motion is, therefore, oscillatory for the range $0<x<x_8$ and is aperiodic for $x \geqslant x_8$.



Graph 4. The phase and group velocities V_w and V_g respectively as functions of the wave number x in case $P=0{\cdot}8$, and $Q = 0{\cdot}05$.

Case (*B*) $P < P^*$

(*i*) When $0 < Q < \phi_E$. In this case also there are four values of x, say x_1', x_2', x_3', x_4' for which $Q = \phi$. The motion is oscillatory for the ranges $x_1' < x < x_2'$ and $x_3' < x < x_4'$. Outside these ranges the motion is aperiodic.

(*ii*) When $\phi_E < Q < 4\,(P-1)^2$. For these ranges of Q there are only two values of x (say x_5' and x_6') for which $Q = \phi$. The oscillations therefore, arise for those values of x for which $x_5' < x < x_6'$. The motion is aperiodic outside this range of x.

(*iii*) When $Q > 4\,(P-1)^2$. In this case there is only one value of x ($= x_7'$) for which $Q = \phi$. Consequently the motion is oscillatory when $0 < x < x_7'$ and aperiodic when $x \geqslant x_7'$.

Section 2. Investigations for the range $P \leqslant \sqrt{3} - 1$, $P > 1$.

(*a*) *Characteristics of the Graph* 2.

From the graph 2 we see that for the range $P \leqslant 3 - 1$, $\phi(x)$ decreases from its initial value $4\,(P-1)^2$ at $x = 0$ till it reaches ϕ_E at $x = x_E$ given by equations (18) and (19) then the curve rises monotonically with x and tends to infinity.

For $P = 1$, we find that $\phi(x)$ increases with x and approaches the line $\phi = 16/3$ asymptotically.

For $P > 1$, we observe from the graph 2 that $\phi(x)$ rises monotonocally with x and tends to infinity as x increases.

(*b*) *Conclusions* :

Case (*A*). $P \leqslant \sqrt{3-1}$

(*i*) When $0 < Q < \phi_E$. According to the criterion (17) the motion is aperiodic since Q is less than ϕ for all values of x. This does not apply to the case $P = \sqrt{3-1}$ since in this case $\phi_E = 0$.

(*ii*) When $\phi_E < Q < 4(P-1)^2$ there are two non-zero values of x, say x_1 and x_2, for which $Q = \phi$. Consequently the oscillations arise only within the wave number range $x_1 < x < x_2$ and the motion is aperiodic elsewhere.

(*iii*) When $Q > 4\ (P-1)^2$. In this case there is only one value of x $(=x_3)$ for which $Q = \phi$. Hence the motion is oscillatory when $0 < x < x_3$ and is aperiodic when $x \geqslant x_3$.

(*Case* (*B*). $P = 1$

(*i*) When $Q < 16/3$. There is one value of x, say x_1', for which $Q = \phi$. Therefore oscillations arise when $0 < x < x_1'$ and aperiodic motion when $x \geqslant x_1'$.

(*ii*) When $Q \geqslant 16/3$. In this case $Q > \phi$ for all values of x. Hence the motion is aperiodic throughout.

Case (*C*). $P > 1$.

(*i*) When $Q \leqslant 4\ (P-1)^2$. Here $Q \leqslant \phi$ for all values of x. Therefore all the modes are damped aperiodically.

(*ii*) When $Q > 4\ (P-1)^2$ there is one value of x $(= x_1'')$ for which $Q = \phi$. The motion is therefore, oscillatory for the wave number range $0 < x < x_1''$ and is aperiodic when $x \geqslant x_1''$.

In table 1, the values of x_E and ϕ_E are listed for different values of P.

S. No.	P	x_E	ϕ_E	S. N.	P	x_E	ϕ_E
1.	·1	·3532	3·2182	6.	·6	1·1890	·2364
2.	·2	·5211	2·5317	7.	·7	1·4273	·0169
3.	·3	·6777	1·7925	8.	$\sqrt{3-1}$	$2/\sqrt[4]{3}$	0
4.	·4	·8315	1·1736	9.	·8	1·7599	·0715
5.	·5	·9977	1·6446	10.	·9	2·3505	·7721

PROPERTIES OF THE MOTION

In the case of aperiodic damping there are two damping coefficients given by

$$-y = \left\{P(4+x^2) + \frac{16+8x^2+3x^4}{4+3x^2}\right\} \pm \left[\left\{P(4+x^2) - \frac{16+8x^2+3x^4}{4+3x^2}\right\}^2 - \frac{4Q(4+x^2)}{4+3x^2}\right]^{\frac{1}{2}} \qquad \ldots\ldots (22)$$

In the case of oscillatory motion there is only one damping coefficient

$$-R(y) = P(4+x^2) + \frac{16+8x^2+3x^4}{4+3x^2} \qquad \ldots\ldots (23)$$

$I(y)$, the angular frequency of oscillations is given by

$$I(y) = \pm\left[\frac{4Q(4+x^2)}{4+3x^2} - \left\{P(4+x^2) - \frac{16+8x^2+3x^4}{4+3x^2}\right\}^2\right]^{\frac{1}{2}} \ldots\ldots (24)$$

These functions are illustrated in Graph 3 for the case (B) (i) (Section 1) (In particular $P = 0{\cdot}8$ and $Q = 0{\cdot}05$).

V_w, the wave velocity and V_g, the group velocity are governed by equations

$$V_w^2 = \{I(y)\}^2/x^2 \qquad \ldots\ldots (25)$$

and

$$V_g^2 = \left\{\frac{d\,I(y)}{dx}\right\}^2$$

$$= \frac{4}{V_w^2}\left[\frac{16Q}{(4+3x^2)} - \left\{P(4+x^2) - \frac{16+8x^2+3x^4}{4+3x^2}\right\}\left\{(P-1) + \frac{32}{(4+3x^2)^2}\right\}\right]^2. \qquad \ldots\ldots (26)$$

Graph 4 gives the variations of V_w and V_g for the case (B) (i) (Section 1) (In particular $P = 0{\cdot}08$ and $Q = 0{\cdot}05$).

REFERENCES

1. Hide, R. *Proc. Camb. Phil. Soc.*, **51** : 179–201, (1955*a*).
2. Hide, R. *Proc. Roy. Soc. London*, **233 A** : 376–396, (1956).
3. Ariel, P. D. and Bhatia, P. K. Waves in a heavy incompressible fluid of finite depth and of variable density I, *Proc. Nat. Inst. Sc. India*, (In Press).

INTRINSIC RELATIONS OF COMPLEX LAMELLAR FLOWS

By

G. PURUSHOTHAM *and* A. INDRASENA

Department of Mathematics, Osmania University, Hyderabad–7, A.P.

[Received on 31st March, 1965]

§ 1. INTRODUCTION

In order to consider the physical and geometrical properties of steady gas flows (thermally non conducting with two specific heats constant in the absence of extraneous forces), the attention of the workers [1–4, 6, 7] in the field of fluid dynamics has been diverted to the application of differintial geometry. Suryanarayan[6] in his recent paper, considering the geometry of the steady, complex lamellar gas flows has further extended the geometric theory of surfaces in the field of fluid mechanics. He has cnnsidered the Beltrami surfaces which are orthogonal to the direction of the flow as surfaces obtained by revolution of a system of confocal hyperbolae. He has experessed only the momentum and continuity relations in intrinsic form, but his investigations in this aspect of study of a complex lamellar gas flows is incomplete. Since he has not considered the variation of the flow quantities along the streamline, principal normal and binormal. Herein we have expressed all the basic equations governing the flow in intrinsic form and definite possible forms for the flow quantities have been derived. Futher it is shown that all the flow quantities including the curvature and torsion of the streamline, mean curvature, Gaussian curvature and Mach number are uniform along the abinormal to the streamline. For homentropic flow the density and the velocity of the fluid are obtained. The compatiability condition when the fluid is incompressible is obtained.

§ 2. BASIC EQUATIONS

(*A*) The basic equations governing an inviscid, steady, thermally nonconducting with two specific heats constant in the absence of extraneous forces in the intrinsic form are given below in the usual notation[5] :

Equation of continuity

$$-J = (K'+K'') = \frac{d}{ds} \log (\varrho q) \tag{1}$$

Momentum equations

$$\frac{1}{2}\frac{dq^2}{ds} = -\frac{1}{\rho}\frac{dp}{ds} \tag{2}$$

$$Kq^2 = -\frac{1}{\rho}\frac{dp}{dn} \tag{3}$$

$$0 = -\frac{1}{\rho}\frac{dp}{db} \tag{4}$$

Energy equation

(5) $$\frac{dS}{ds} = 0$$

Crocco's Vorticity relations

(6) $$-q\left(Kq - \frac{dq}{dn}\right) = \frac{d}{dn}(h + q^2/2) - T\frac{dS}{dn}$$

(7) $$q\frac{dq}{db} = \frac{d}{db}(h + q^2/2) - T\frac{dS}{db}$$

(8) $$\frac{d}{ds}(h + q^2/2) = 0$$

using (1) – (4) and the equation of state we have

(9) $$-J = (K' + K'') = (M^2 - 1)\frac{d}{ds}\log(q) = \frac{(1 - M^2)}{\rho q^2}\frac{dp}{ds}$$

(10) $$\frac{dp}{dn} = -KM^2\rho + \frac{\partial p}{\partial S}\frac{dS}{dn}$$

(11) $$-KM^2 = \frac{d}{dn}\log\rho$$

(12) $$\frac{dS}{db} = \left(\frac{\partial p}{\partial S}\right)^{-1}$$

For Prim gas (10), (12) become

(13) $$\frac{dS}{dn} = -\gamma Jev\left[KM^2 + \frac{d}{dn}\log\rho\right]$$

(14) $$\frac{dS}{db} = -\gamma Jev\left[\frac{d}{db}\log\rho\right]$$

By means of cross differentiation of (13) and (14) we obtain

(15) $$\frac{d}{db}(KM^2) = 0$$

using equations of momentum and state we have

(16) $$\frac{1}{\rho}\frac{dp}{ds} = \frac{dh}{ds} = -q\frac{dq}{ds} = \frac{c^2}{\rho}\frac{d\rho}{ds}$$

(17) $$\frac{dh}{dn} = -Kq^2 + T\left[K\rho M^2 + \frac{d\rho}{dn}\right]\left(\frac{\partial p}{\partial S}\right)^{-1}$$

(18) $$\frac{dh}{dn} = -\left[KM^2(c^2 + T\gamma Jev) + T\gamma Jev\frac{d}{dn}\log\rho\right]$$

(19) $$\frac{\partial \rho}{\partial S} \frac{dh}{db} = T \frac{d\rho}{db}$$

(20) $$\frac{dT}{ds} = -\frac{1}{\rho^2} \frac{\partial \rho}{\partial S} \frac{dp}{ds} = \frac{dp}{ds} \frac{d}{ds} \log \left(\frac{1}{\rho} \right)$$

((21) $$\frac{dT}{ds} / \frac{\partial}{\partial S} \left(\frac{1}{\rho} \right) = \frac{dp}{ds} = \rho \frac{dh}{ds} = -q\rho \frac{dq}{ds} = c^2 \frac{d\rho}{ds}$$

(B) Beltrami Surfaces. Considering the Beltrami surfaces to be surfaces of revolution obtained by revolving a family of hyperbolas and transforming the intrinsic relations given above, we shall study the properties of flows.

Geometrical relations.—The surfaces of revoluations when they are family of hyperbolas are defined as

(22) $$x = u \cos \theta \qquad y = u \sin \theta \qquad \alpha u^2 - \beta z^2 = \delta$$

where α, β are constants to be determined later and 'δ' is parameter.

The normals to the hyperbolas defined in the meridian plane $\alpha u^2 - \beta z^2 = \delta$ are the tangents to the streamlines which determine the direction of the flow in meridian plane. Therefore the principal normal vector of the streamline is along the tangent to the hyperbolas, and the binormal is along the parallels perpendicular to the meridian plane—

Considering $\vec{t}$, $\vec{n}$ and $\vec{b}$ as the unit tangent, principal normal and binormal to the streamline in the present problem these correspond to

(23) $$\vec{t} = \vec{i_u} \frac{u\alpha}{t} - \vec{i_z} \frac{\beta z}{t}$$

(24) $$\vec{n} = \vec{i_u} \frac{\beta z}{t} + \vec{i_z} \frac{u\alpha}{t}$$

(25) $$\vec{b} = \vec{i_\theta}$$

Where $$t^2 = \alpha^2 u^2 + \beta^2 z^2$$

§ 3. INTERINSIC RELATIONS

Using (23) the mean curvature and curvature of the streamline respectively are

(26) $$J = \frac{(2\alpha - \beta)}{t} - \frac{1}{t^3} (\alpha^3 u^2 - \beta^3 z^2)$$

(27) $$K = \frac{\alpha \beta \delta}{t^3}$$

Therefore the equation (1) becomes

$$(28) \qquad J = \frac{(2\alpha - \beta)}{t} - \frac{1}{t^3}(\alpha^3 u^2 - \beta^3 z^2) = \frac{\alpha u}{t}\frac{\partial}{\partial u}\log(\rho q) - \frac{\beta z}{t}\frac{\partial}{\partial z}\log(\rho q)$$

From this writing down the Lagrange's equations

$$(29) \qquad \frac{du}{\alpha u} = \frac{dz}{-\beta z} = \frac{(d \log \rho q)}{(2\alpha - \beta) - \frac{1}{t^2}(\alpha 3\, u^2 - \beta^3 z^2)}$$

An intermediate integral is

$$(30) \qquad u^{\beta}\ z^{\alpha} = c \text{ (constant)}$$

using (30) in (29) we obtain

$$(31) \qquad \log(\rho q) = -\frac{1}{\alpha}\int \frac{1}{t^2}(\alpha^3 u^3 - \beta^3 z^2)\, du + \chi(u^{\beta} x^{\alpha}) + \frac{1}{\alpha}\int \frac{(2\alpha - \beta)}{u\, t^2}\, du$$

Equation (30) is a purely geometrical relation to be satisfied between co-ordinate lines. (31) Gives us the relation between the density of the fluid and the magnitude of the velocity vector.

Equation (5) simplifies to

$$(32) \qquad u\alpha \frac{\partial S}{\partial u} - \beta z \frac{\partial S}{\partial z} = 0$$

Again writing down the Lagrange's equations we obtain

$$(33) \qquad S = S(u^{\beta} z^{\alpha}) = S\{(x^2 + y^2)^{\beta/2} z^{\alpha}\}$$

This shows that

$$(34) \qquad \frac{\partial S}{\partial \theta} = 0$$

Equations (2), (3), (4) respectively reduce to

$$(35) \qquad \frac{1}{2}\left[\alpha u \frac{\partial q^2}{\partial u} - \beta z \frac{\partial q^2}{\partial z}\right] = -\frac{1}{\rho}\left[\alpha u \frac{\partial p}{\partial u} - \beta z \frac{\partial p}{\partial z}\right]$$

$$(36) \qquad \frac{2}{q}\frac{\alpha\beta\delta}{t^2} = -\frac{1}{\rho}\left[\beta z \frac{\partial p}{\partial u} + \alpha u \frac{\partial p}{\partial x}\right]$$

$$(37) \qquad \frac{\partial p}{\partial \theta} = 0$$

Using the equation of state (34) and (37) we obtain

$$(38) \qquad \frac{\partial \rho}{\partial \theta} = 0$$

Differentiating (3) with respect to 'θ' and using (37) and (38) we obtain

(39) $$\frac{\partial \eta}{\partial \theta} = 0$$

Using (43), (38) and (39) we get

(40) $$\frac{h\partial}{\partial \eta} = 0$$

Defferentiating (9) with respect to 'θ' and using the above relations we have

(41) $$\frac{\partial M}{\partial \theta} = 0$$

Differentiating (17) with respect to 'θ' we obtain

(42) $$\frac{\partial T}{\partial \theta} = 0$$

using (41) in (15) we obtain

(43) $$\frac{\partial K}{\partial \theta} = 0$$

Following Weatherburn[8] We have the relation

(44) $$\alpha(\tau^2 + K^2) = K$$

using (27) in (44) we obtain

(45) $$\tau^2 = \frac{\alpha\beta\delta}{t^3}\left(\frac{1}{\alpha} - \frac{\alpha\beta\delta}{t^3}\right)$$

where 'τ' is the torsion of the streamline

Differentiating (44) with respect to 'θ' and using (43) We get

(46) $$\frac{\partial \tau}{\partial \theta} = 0$$

From the above relations it follows that for a Betrami surfaces to be surfaces of revolution obtained by a family of hyperbolas to exist in the gas flow described above flow quantities are uniform in the Osculating plane

From the theory of surfaces, the Gaussian curvature K is given by[8]

(47) $$2K = \operatorname{div}\left(\vec{t}\,\operatorname{div}\vec{t} + \vec{t}\wedge\operatorname{curl}\vec{t}\right)$$

using (1) and (27) in (47) we obtain

(48) $$2K = \left[\frac{d}{ds} \log (\rho q) \right]^2 + \frac{\alpha^2\beta^2\delta^2}{t^6} - \frac{d^2 \log (\rho q)}{ds^2} - \frac{d}{du} \left(\frac{\alpha\beta\delta}{t^3} \right)$$

which is independant of 'θ' and in current coordinate system this reduces to

(49) $$2K = A^2 + \frac{\alpha^2\beta^2\delta^2}{t^6} - \left(\frac{\alpha u}{t} \frac{\partial A}{\partial u} - \frac{\beta z}{t} \frac{\partial A}{\partial z} \right) - \left[\frac{\beta z}{t} \frac{\partial}{\partial z} \left(\frac{\alpha\beta\delta}{t^3} \right) + \frac{\alpha u}{t} \frac{\partial}{\partial z} \left(\frac{\alpha\beta\delta}{t^3} \right) \right]$$

Where $A = \frac{\alpha u}{t} \frac{\partial}{\partial u} (\log \rho q) - \frac{\beta z}{t} \frac{\partial}{\partial z} \log (\rho q)$

From Crocco's vorticity equation (6) we have

(50) $$\text{curl } \vec{q} \cdot \vec{b} = -\frac{1}{q} \left[\frac{\beta z}{t} \left\{ \frac{\partial B}{\partial u} - \frac{T \partial S}{\partial u} \right\} + \frac{\alpha u}{t} \left\{ \frac{\partial B}{\partial z} - \frac{T \partial S}{\partial z} \right\} \right]$$

Which gives the vorticity component along 'θ' direction

The relation (10) simplifies to

(51) $$\left(\beta z \frac{\partial \rho}{\partial u} + \alpha u \frac{\partial \rho}{\partial z} \right) = -\frac{\alpha\beta\delta\rho M^2}{t^2} + \frac{\partial \rho}{\partial S} S' \left\{ \beta^2 u^{\beta-2} + \alpha^2 z^{\alpha-2} \right\}_{uz}$$

For Primgas (51) simplifies to

(52) $$\beta z \frac{\partial}{\partial u} \log \rho + \alpha u \frac{\partial}{\partial z} \log \rho = -\frac{\alpha\beta M^2\delta}{t^2} - S' \left[\beta^2 u^{\beta-2} + \alpha^2 z^{\alpha-2} \right]_{uz}$$

Where $S' = dS \left(u^\beta \; z^\alpha \right)$

When the flow is homentrophic (51) and (52) reduce to

(53) $$\beta z \frac{\partial}{\partial u} \log \rho + \alpha u \frac{\partial}{\partial u} \log \rho = -\frac{\alpha\beta\delta M^2}{t^2}$$

writing down the Lagrange's equations for (53) we have

(54) $$\frac{du}{\alpha u} = \frac{dz}{\beta z} = -\frac{t^2 \, d \log \rho}{\alpha\beta\delta \, m^2}$$

An intermediate integral of (54) is

(55) $$\alpha u^2 - \beta z^2 = \text{const}$$

Hence

(56) $$\log \rho = f\,(\alpha u^2 - \beta z^2) - \alpha\delta \int \frac{M^2 \, du}{z \, t^2}$$

Using (31) and (56) we obtain

$$\log q = \alpha S \int \frac{M^2}{z}\frac{du}{t^2} - f(\alpha u^2 - \beta z^2) - \frac{1}{\alpha}\int \frac{(\alpha^3 u^2 - \beta^3 z^2)}{t^2}\, du + X\left(4^{\beta}\ z^{\alpha}\right) + \frac{1}{\alpha}\int \frac{(2\alpha - \beta)}{u\ t^2}\, ds \tag{57}$$

Which defines the velocity of the fluid particles when the flow is homentrophic.

Equation (7), (12), (14) and (19) are identically satisfied. The equation (8) can be written in intrinsic form as

$$\alpha u \frac{\partial B}{\partial u} - \beta z \frac{\partial B}{\partial z} = 0 \tag{58}$$

Writing the Lagrange's equations we obtain

$$B = B\ \left\{ (x^2 - y^2)^{\beta/2}\ z^{\alpha} \right\} \tag{59}$$

Equations (11), (16), (17), 18), (20) and (21) respectively in the intrinsic form are given as

$$-\frac{M\delta^2}{t} = z/\alpha\ \frac{\partial \log \rho}{\partial u} + u/\beta\ \frac{\partial}{\partial z}\log \rho \tag{60}$$

$$\frac{1}{\rho}\left(\alpha u\frac{\partial p}{\partial u} - \beta z\frac{\partial p}{\partial z}\right) = \left(\alpha u\frac{\partial h}{\partial u} - \beta z\frac{\partial h}{\partial z}\right) = -q\left(\alpha u\frac{\partial q}{\partial u} - \beta z\frac{\partial q}{\partial z}\right) = \frac{c^2}{\rho}\left(\alpha u\frac{\partial \rho}{\partial u} - \beta z\frac{\partial \rho}{\partial z}\right) \tag{61}$$

$$\left(\beta z\frac{\partial h}{\partial u} + u\alpha\frac{\partial h}{\partial z}\right) = -\frac{\alpha\beta\delta q^2}{t^2} + T\left[\frac{\alpha\beta\delta\rho M^2}{t^2} + \left(\beta z\frac{\partial \rho}{\partial u} + u\alpha\frac{\partial \rho}{\partial z}\right)\right]\left(\frac{\partial \rho}{\partial S}\right)^{-1} \tag{62}$$

$$\left(\beta z\frac{\partial h}{\partial u} + u\alpha\frac{\partial h}{\partial z}\right) = -\left[\frac{\alpha\beta\delta}{t^2}\ M^2 (c^2 + T\gamma Jcv) + T\gamma Jcv\left(\beta z\frac{\partial}{\partial u}\log \rho + u\alpha\frac{\partial}{\partial z}\log \rho\right)\right] \tag{63}$$

$$\left(\beta z\frac{\partial T}{\partial u} + u\alpha\frac{\partial T}{\partial z}\right) = -\rho^{\frac{1}{2}}\left(\frac{\partial \rho}{\partial S}\right)\left(\alpha u\frac{\partial p}{\partial u} - \beta z\frac{\partial p}{\partial z}\right) = \frac{1}{t}\left(\alpha u\frac{\partial p}{\partial u} - \beta z\frac{\partial p}{\partial z}\right)\left[\alpha u\frac{\partial}{\partial u}\left(\frac{1}{\rho}\right) - \beta z\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right] \tag{64}$$

$$\left(\alpha u\frac{\partial T}{\partial u} - \beta z\frac{\partial T}{\partial z}\right)\left[\frac{\partial}{\partial S}\left(\frac{1}{\rho}\right)\right]^{-1} = \left(\alpha u\frac{\partial p}{\partial u} - \beta z\frac{\partial p}{\partial z}\right) = \rho\left(\alpha u\frac{\partial h}{\partial u} - \beta z\frac{\partial h}{\partial z}\right) = c^2\left(\alpha u\frac{\partial \rho}{\partial z} - \beta z\frac{\partial \rho}{\partial z}\right) \tag{65}$$

These are the intrinsic relations to be satisfied by a steady inviscid, thermally non-conducting gas flows.

For the incompressible flow the equations (1), (2), (3) and (4) give

$$\frac{d}{dn}\log J + \frac{1}{J}\frac{dK}{ds} + 2J\frac{d}{du}\log q - 2K = 0 \tag{66}$$

This datermines the relations between the velocity of the fluid particles, curvature and mean curvature of the streamline in intrinsic form which in the current coordinate system is

$$\frac{\beta z}{t}\frac{\partial}{\partial u}\log m + \frac{u\alpha}{t}\frac{\partial}{\partial z}\log m + \frac{1}{m}\left\{\frac{\alpha u}{t}\frac{\partial}{\partial u}\left(\frac{\alpha\beta\delta}{t^3}\right) - \frac{\beta z}{t}\frac{\partial}{\partial z}\left(\frac{\alpha\beta\delta}{t^3}\right)\right\} + 2m\left\{\frac{\beta z}{t}\frac{\partial}{\partial u}\log q + \frac{\alpha u}{t}\frac{\partial}{\partial z}\log q\right\} - \frac{2\alpha\beta\delta}{t^3} = 0 \tag{67}$$

Where $$m = \frac{\alpha u}{t}\frac{\partial}{\partial u}\log q + \frac{\beta z}{t}\frac{\partial}{\partial z}\log q$$

REFERENCES

1. R. P. Kanwal. *Jr. Maths. Mech.*, **6** : 521, (1957).
2. R. P. Kanwal. *ZAMM* **41** : S. 462, p. 521, (1961).
3. Prem Kumar. *Tensor* (N. S.) **12** : 239, (1962).
4. Prem Kumar. *Tensor* (N. S.) **12** : 65, (1962).
5. G. Purushotham. Doctoral Thesis presented to Osmania University.
6. E. R. Suryanarayan. *Jr. Maths. and Mech.* **13** : 163, (1964).
7. C. Truesdell. *ZAMM*. B. **40**, S. 9, (1960).
8. C. E. Weatherburn. Diff. Geom. of three dim. p. 15, Camb. Univ. Press, (1955).

INTEGRALS INVOLVING APPELL'S FUNCTIONS

By

R. K. SAXENA *and* B. L. SHARMA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 7th April, 1965]

ABSTRACT

In this paper certain integrals involving Appell's functions ha e been evaluated in terms of hypergeometric functions of three variables with the help of certain theorems of integral transforms given by Saxena [2, p. 132] and Rathie [2, p. 132].

1. Introductory. Varma [6, p. 209] has given a generalization of the classical Laplace's integral

$$f(p) = p \int_0^\infty e^{-pt} h(t)\, dt \qquad \ldots. (1)$$

in the form

$$\phi(p) = p \int_0^\infty (pt)^{\mu - \frac{1}{2}} e^{-\frac{1}{2}pt} W_{k,\mu}(pt)\, h(t)\, dt. \qquad \ldots. (2)$$

In what follows (1) and (2) will be denoted by $f(p) \doteqdot h(t)$ and $\phi(p)$

$\overset{v}{\underset{k,\mu}{=}} h(t)$ respectively.

For the definitions and properties of hypergeometric functions of three variables, *see* Saran (**4**, pp. 77–91).

The following results can be easily derived from Saran's formula (**5**, p. 134)

$$t^{\lambda-1} k_\mu(xt)\, \psi_2(b_2; c_2, c_3; 2yt, 2zt)$$

$$\doteqdot p \sum_{\mu,-\mu} 2^{-\mu-1} x^{\mu} (x+p)^{-\lambda-\mu}\, \Gamma(-\mu)\, \Gamma(\lambda+\mu)$$

$$F_E\left[\lambda+\mu, \lambda+\mu, \lambda+\mu, b_2, b_2; 1+2\mu, c_2, c_3; \frac{2x}{p+x}, \frac{2y}{p+x}, \frac{2z}{p+x}\right], \qquad \ldots. (3)$$

valid for

$$R(\lambda \pm \mu) > 0,\ R(p+x) > 2\, R(\sqrt{y} + \sqrt{z})^2.$$

$$t^{\lambda-1} k_\mu(xt)\, \phi_2(b_2, b_3; c_2; 2yt, 2zt)$$

$$\doteqdot p \sum_{\mu,-\mu} 2^{-1-\mu} x^{\mu} (p+x)^{-\lambda-\mu} \Gamma(-\mu)\, \Gamma(\lambda+\mu)$$

$$F_G\left(\lambda+\mu, \lambda+\mu, \lambda+\mu, \tfrac{1}{2}+\mu, b_2, b_3; 1+2\mu, c_2, c_2; \frac{2x}{p+x}, \frac{2y}{p+x}, \frac{2z}{p+x}\right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(p + x - 2y - 2z) > 0$. .... (4)

$$t^{\lambda-2}\, k_\mu\,(yt)\ \psi_1\,(b_1, a_1\,;\, c_1, c_3\,;\, 2x,\, 2zt)$$

$$\doteqdot p \sum_{\mu, -\mu} 2^{-1-\mu}\, y^\mu\, (p + y)^{-\lambda-\mu}\, \Gamma(-\mu)\ \Gamma(\lambda + \mu)$$

$$F_K\left(a_1, \lambda + \mu, \lambda + \mu, b_1, \tfrac{1}{2} + \mu, b_1\,;\, c_1, 2\mu + 1, c_3\,;\, 2x,\ \frac{2y}{p + y}, \frac{2z}{p + y} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R[(p+y)(1 - 2x)] > R(2z)$. .... (5)

$$t^{\lambda-1}\, k_\mu\,(xt)\ \phi_1\,(a_1, b_2\,;\, c_2\,;\, 2zt,\, 2y)$$

$$\doteqdot p \sum_{\mu, -\mu} 2^{-\mu-1}\, x^\mu\, (p+x)^{-\lambda-\mu}\ \Gamma(-\mu)\ \Gamma(\lambda + \mu)$$

$$F_M\left(\tfrac{1}{2} + \mu, a_2, a_2, \lambda + \mu, b_2, \lambda + \mu\,;\, 1 + 2\mu, c_2, c_2\,;\, \frac{2x}{p + z}, 2y,\ \frac{2z}{p + x} \right)$$

valid for $(\lambda R \pm \mu) > 0$, $R(p + x) > R(2z)$. .... (6)

$$t^{\lambda-1}\, k_\mu\,(xt)\ \Xi_1\,(a_2, a_3, b_2, c_2\,;\, 2y,\, 2zt) \doteqdot$$

$$p \sum_{\mu, -\mu} 2^{-1-\mu} x^\mu\, (p + x)^{-\mu-\lambda}\ \Gamma(-\mu)\ \Gamma(\lambda + \mu)$$

$$F_N\left(\mu + \tfrac{1}{2}, a_2, a_3, \lambda + \mu, b_2, \lambda + \mu\,;\, 1 + 2\mu, c_2\ c_2\,;\ \frac{2x}{p + x}, 2y,\ \frac{2z}{p + x} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(p + x) > R(2z)$. .... (7)

$$t^{\lambda-\mu-1}\ e^{-at}\ \psi_2\,(b_2\,;\, c_2, c_3\,;\, yt,\, zt)$$

$$\frac{\nu}{k, \mu}\ p^{\mu+1} \sum_{\mu, -\mu} \frac{\Gamma(-2\mu)\quad \Gamma(\lambda + \mu)}{\Gamma(\tfrac{1}{2} - \mu - k)\ \alpha^{\lambda+\mu}}\ p^\mu$$

$$F_E\left(\lambda + \mu, \lambda + \mu, \lambda + \mu, \tfrac{1}{2} + k + \mu, b_2, b_2\,;\, 1 + 2\mu, c_2, c_3\,;\, -\frac{p}{\alpha}, \frac{y}{\alpha}, \frac{z}{a} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(p + \alpha) > R(\sqrt{y} + \sqrt{z})^2$. .... (8)

$$t^{\lambda-\mu-1}\ e^{-at}\quad \phi_2\,(b_2, b_3\,;\, c_2\,;\, yt,\, zt)$$

$$\frac{\nu}{k, \mu}\ p^{\mu+1} \sum_{\mu, -\mu} \frac{\Gamma(-2\mu)\quad \Gamma(\lambda + \mu)}{\Gamma(\tfrac{1}{2} - \mu - k)\ \alpha^{\lambda+\mu}}\ p^\mu$$

$$F_G\left(\lambda + \mu, \lambda + \mu, \lambda + \mu, \tfrac{1}{2} + k + \mu, b_2, b_3\,;\, 1 + 2\mu, c_2, c_2\,;\, -\frac{p}{\alpha}, \frac{y}{a}, \frac{z}{\alpha} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(p + \alpha) > R(y + z)$. .... (9)

$$t^{\lambda-\mu-1} \; e^{-\alpha t} \; \psi_1 (b_1, a_1, c_1, c_3 \,;\, x, zt)$$

$$\frac{\nu}{\overline{k.\mu}} \; p^{\mu+1} \sum_{\mu, -\mu} \frac{\Gamma(-2\mu) \quad \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k) \; \alpha^{\lambda+\mu}} \; p^{\mu}$$

$$F_K \left(a_1, \lambda+\mu, \lambda+\mu, b_1, \tfrac{1}{2}+\mu+k, b_1 \,;\, c_1, 1+2\mu, c_3 \,;\, x, -\frac{p}{\alpha} \;, \frac{z}{\alpha} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R\,(p+\alpha) > R\left(\frac{z}{1-z}\right)$. .... (10)

$$t^{\lambda-\mu-1} \; e^{-\alpha t} \; \phi_1 (a_1, b_2 \,;\, c_2 \,;\, zt, y) \; \frac{\nu}{\overline{k,\mu}}$$

$$p^{\mu+1} \sum_{\mu, -\mu} \frac{\Gamma(-2\mu) \quad \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k) \; \alpha^{\lambda+\mu}} \, p^{\mu}$$

$$F_M \left(\tfrac{1}{2}+k+\mu, a_2, a_2, \lambda+\mu, b_2, \lambda+\mu \,;\, 1+2\mu, c_2, c_2 \,;\, -\frac{p}{\alpha} \,,\, y, \frac{z}{\alpha} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(p+\alpha) > R(z)$. .... (11)

$$t^{\lambda-\mu-1} \; e^{-\alpha t} \; \Xi_1(a_2, a_3, b_2, c_2 \,;\, y, zt)$$

$$\frac{\nu}{\overline{k,\mu}} \; p^{1+\mu} \sum_{\mu, -\mu} \frac{\Gamma(-2\mu) \qquad \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k) \; \alpha^{\lambda+\mu}} \, p^{\mu}$$

$$F_N \left(\tfrac{1}{2}+k+\mu, a_2, a_3, \lambda+\mu, b_2, \lambda+\mu \,;\, 1+2\mu, c_2, c_2 \,;\, -\frac{p}{\alpha} \,, y, \frac{z}{a} \right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(p+z) > R(a)$. .... (12)

2.

Section A

In this section, we obtain some integrals with the help of the following theorem due to Saxena [3, p, 154].

Theorem :—

If $\quad \phi\,(p) \doteqdot h\,(t)$

and $\quad L\,(\nu, p, x) \doteqdot k_\nu\,(xt)\; h\,(t)$

then

$$\int_0^\infty t^{-\nu}\,(a+bt+ct^2)^{-1} \phi\left(\frac{a+bt+ct^2}{t}\right) dt = \frac{2}{b} \left(\frac{c}{a}\right)^{\frac{v}{2}} L\,(v, b, 2\sqrt{ac}\,) \quad \text{.... (13)}$$

provided that the integrals are absolutely convergent, $R(a) > 0$, $R(c) > 0$, and $h\,(t)$ is independant of x.

We take Erdélyi [1, p. 223]

$$h(t) = t^{\lambda-1}\, \psi_2(b_2 ; c_2, c_3 ; 2y\, t, 2zt)$$

$$\doteqdot \Gamma(\lambda)\, p^{1-\lambda}\, F_4\left(\lambda, b_2 ; c_2, c_3 ; \frac{2y}{p}, \frac{2z}{p}\right)$$

$$= \phi(p),$$

$$R(\lambda) > 0,\ R(p) > 0,\ R(2y) > R(2z) \qquad \ldots. (14)$$

Also from (3),

$$k_\nu(xt)\, h(t) = t^{\lambda-1}\, k_\nu(xt)\, \psi_2(b_2 ; c_2, c_3 ; 2yt, 2zt)$$

$$\doteqdot p \sum_{\nu, -\nu} 2^{-1-\nu} x^\nu (p+x)^{-\lambda-\nu}\, \Gamma(-\nu)\, \Gamma(\lambda+\nu)$$

$$F_E\left(\lambda+\nu, \lambda+\nu, \lambda+\nu, \tfrac{1}{2}+\nu, b_2, b_2 ; 1+2\nu, c_2, c_3 ; \frac{2x}{p+x}, \frac{2y}{p+x}, \frac{2z}{p+x}\right)$$

$$= L(v, p, x),\ R(\lambda \pm \nu) > 0,\ R(p+x) > 2\, R(\sqrt{y} + \sqrt{z})^2. \qquad \ldots. (15)$$

Using (14) and (15) in (13), we get.

$$\int_0^\infty t^{\lambda-\nu-1} (a+bt+ct^2)^{-\lambda}\, F_4\left(\lambda, b_2 ; c_2, c_3 ; \frac{2yt}{a+bt+ct^2}, \frac{2zt}{a+bt+ct^2}\right) dt$$

$$= \left(\frac{c}{a}\right)^{\frac{v}{2}} \sum_{\nu, -\nu} \frac{\Gamma(-\nu)\, \Gamma(\lambda+\nu)}{\Gamma(\lambda)} (ac)^{\frac{\nu}{2}} (b + 2\sqrt{ac})^{-\lambda-\nu}$$

$$F_E\left(\lambda+\nu, \lambda+\nu, \lambda+\nu, \tfrac{1}{2}+\nu, b_2, b_2 ; 1+2\nu, c_2, c_3 ; \frac{4\sqrt{ac}}{b+2\sqrt{ac}}, \frac{2y}{b+2\sqrt{ac}}, \frac{2z}{b+2\sqrt{ac}}\right)$$

valid for $R(\lambda \pm \nu) > 0,\ a, b, c, y, z > 0,\ (\sqrt{y} + \sqrt{z})^2 < \frac{b}{2} - \sqrt{ac}.$ $\qquad \ldots. (16)$

In particular, if $y \to 0$, then $F_E \to F_2$, we obtain from (16)

$$\int_0^\infty t^{\lambda-\nu-1} (a+bt+ct^2)^{-\lambda}\, {}_2F_1\left(\lambda, b_2 ; c_3 ; \frac{2zt}{a+bt+ct^2}\right) dt$$

$$= \left(\frac{c}{a}\right)^{\frac{v}{2}} \sum_{\nu, -\nu} \frac{\Gamma(-\nu)\quad \Gamma(\lambda+\nu)}{\Gamma(\lambda)} (ac)^{\frac{\nu}{2}} (b + 2\sqrt{ac})^{-\lambda-\nu}$$

$$F_2\left(\lambda+\nu, \tfrac{1}{2}+\nu, b_2 ; 1+2\nu, c_3 ; \frac{4\sqrt{ac}}{b+2\sqrt{ac}}, \frac{2y}{b+2\sqrt{ac}}\right)$$

valid for $R(\lambda \pm \nu) > 0,\ a, b, c, y > 0, y < \frac{b}{2} - \sqrt{ac}.$ $\qquad \ldots. (17)$

In a similar way, we obtain the following integrals by using the results (4) to (7) and Erdélyi [1, 222–223]

$$\int_0^\infty t^{\lambda-\nu-1}(a+bt+ct^2)^{-\lambda} F_1\left(\lambda, b_2, b_3; c_2; \frac{2yt}{a+bt+ct^2}, \frac{2zt}{a+bt+ct^2}\right) dt$$

$$= \left(\frac{c}{a}\right)^{\frac{\nu}{2}} \sum_{\nu, -\nu} \frac{\Gamma(-\nu)\ \Gamma(\lambda+\nu)}{\Gamma(\lambda)} (ac)^{\frac{\nu}{2}} (b+2\sqrt{ac})^{-\lambda-\nu}$$

$$F_G\left(\lambda+\nu, \lambda+\nu, \lambda+\nu, \tfrac{1}{2}+\nu, b_2, b_3; 1+2\nu, c_2, c_2; \frac{4\sqrt{ac}}{b+2\sqrt{ac}}, \frac{2y}{b+2\sqrt{ac}}, \frac{2z}{b+2\sqrt{ac}}\right)$$

valid for $R(\lambda \pm \nu) > 0, a, b, c, y, z > 0, y < \frac{b}{2} - \sqrt{ac}, z < \frac{b}{2} - \sqrt{ac}$. .... (18)

$$\int_0^\infty t^{\lambda-\mu-1}(a+bt+ct^2)^{-\lambda} F_2\left(b_1, a_1, \lambda; c_1, c_3; 2x, \frac{2zt}{a+bt+ct^2}\right) dt$$

$$= \left(\frac{c}{a}\right)^{\frac{\mu}{2}} \sum_{\mu, -\mu} \frac{\Gamma(-\mu)\ \Gamma(\lambda+\mu)}{\Gamma(\lambda)} (ac)^{\frac{\mu}{2}} (b+2\sqrt{ac})^{-\lambda-\mu}$$

$$F_K\left(a_1, \lambda+\mu, \lambda+\mu, b_1, \tfrac{1}{2}+\mu, b_1; c_1, 1+2\mu, c_3; 2x, \frac{4\sqrt{ac}}{b+2\sqrt{ac}}, \frac{2z}{b+2\sqrt{ac}}\right)$$

valid for $R(\lambda \pm \mu) > 0, a, b, c, x, z > 0, 2z < (1-2x)(b-2\sqrt{ac})$. .... (19)

$$\int_0^\infty t^{\lambda-\mu-1}(a+bt+ct^2)^{-\lambda} F_1\left(a_1, \lambda, b_2; c_2; \frac{2zt}{a+bt+ct^2}, 2y\right) dt$$

$$= \left(\frac{c}{a}\right)^{\frac{\mu}{2}} \sum_{\mu, -\mu} \frac{\Gamma(-\mu)\ \Gamma(\lambda+\mu)}{\Gamma(\lambda)} (ac)^{\frac{\mu}{2}} (b+2\sqrt{ac})^{-\lambda-\mu}$$

$$F_M\left(\mu+\tfrac{1}{2}, a_2, a_2, \lambda+\mu, b_2, \lambda+\mu; 1+2\mu, c_2, c_2; \frac{4\sqrt{ac}}{b+2\sqrt{ac}}, 2y, \frac{2z}{b+2\sqrt{ac}}\right)$$

valid for $R(\lambda \pm \mu) > 0, a, b, c, y, z > 0, z < \frac{b}{2} - \sqrt{ac}, y < \frac{1}{2}$. .... (20)

$$\int_0^\infty t^{\lambda-\mu-1}(a+bt+ct^2)^{-\lambda} F_3\left(a_2, a_3, b_2, \lambda; c_2; 2y, \frac{2zt}{a+bt+ct^2}\right) dt$$

$$= \left(\frac{c}{a}\right)^{\frac{\mu}{2}} \sum_{\mu, -\mu} \frac{\Gamma(-\mu)\ \Gamma(\lambda+\mu)}{\Gamma(\lambda)} (ac)^{\frac{\mu}{2}} (b+2\sqrt{ac})^{-\lambda-\mu}$$

$$F_N\left(\tfrac{1}{2}+\mu, a_2, a_3, \lambda+\mu, b_2, \lambda+\mu; 1+2\mu, c_2, c_2; \frac{4\sqrt{ac}}{b+2\sqrt{ac}}, 2y, \frac{2z}{b+2\sqrt{ac}}\right)$$

valid for $R(\lambda \pm \mu) > 0, a, b, c, y, z > 0, y(b-2\sqrt{ac}) + z(1-2y) < 0$. .... (21)

3. Section B

In the last, we obtain some integrals involving the product of hypergeometric function and Appell's functions with the help of the following theorem due to Rathie [2, p. 132].

Theorem :—

If $\phi(p) \overset{\nu}{\underset{k,\mu}{=}} h(t)$

and $f(p) \doteqdot t^l\ h(t)$

then

$$\phi(p) = \frac{p^{\frac{1}{2}+\mu+k}}{\Gamma(\frac{1}{2}+l-\mu-k)} \int_0^\infty t^{l-\mu-k-\frac{1}{2}} (t+p)^{-1}$$

$${}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu ; \tfrac{1}{2}+l-k-\mu ; -\frac{t}{p}\right) f(t+p)\, dt, \qquad \ldots . (22)$$

provided that $R(p) > 0$, $R(\frac{1}{2}+l-\mu-k) > 0$ and the integral is convergent.

From (8), we have

$$h(t) = t^{\lambda-\mu-1} e^{-\alpha t} \psi_2(b_2 ; c_2, c_3 ; yt, zt)$$

$$\overset{\nu}{\underset{k,\mu}{=}} p^{1+\mu} \sum_{\mu, -\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} \frac{\Gamma(\lambda+\mu)}{\alpha^{\lambda+\mu}} p^{\mu}$$

$$F_E\left(\lambda+\mu, \lambda+\mu, \tfrac{1}{2}+k+\mu, b_2, b_2 ; 1+2\mu, c_2, c_3 ; -\frac{p}{\alpha}, \frac{y}{\alpha}, \frac{z}{\alpha}\right) \qquad \ldots . (23)$$

$$= \phi(p).$$

Erdélyi [1, p. 223]

$$t^l\ h(t) = t^{l+\lambda-\mu-1} e^{-\alpha t} \psi_2(b_2 : c_2, c_3 ; yt, zt)$$

$$\doteqdot p\ \Gamma(l+\lambda-\mu)(p+\alpha)^{\mu-l-\lambda}$$

$$F_4\left(l+\lambda-\mu, b_2 ; c_2, c_3 ; \frac{y}{p+\alpha}, \frac{z}{p+\alpha}\right)$$

$$= f(p),\ R(p+\alpha) > 0,\ R(y), R(z)\ R(l+\lambda-\mu) > 0. \qquad \ldots . (24)$$

Using (23) and (24) in (22), we obtain

$$\int_0^\infty t^{l-\mu-k-\frac{1}{2}} (t+p+\alpha)^{\mu-l-\lambda}\ {}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu ;\right.$$

$$\left. \tfrac{1}{2}+l-k-\mu ; -\frac{t}{p}\right)$$

$$F_4\left(l+\lambda-\mu, b_2; c_2, c_3; \frac{y}{t+p+\alpha}, \frac{z}{t+p+\alpha}\right) dt =$$

$$\frac{\Gamma(\frac{1}{2}+l-\mu-k)}{\Gamma(\lambda+l-\mu)} p^{\frac{1}{2}-k} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)\quad \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k)\ \alpha^{\lambda+\mu}} p^{\mu}$$

$$F_E\left(\lambda+\mu, \lambda+\mu, \lambda+\mu, \tfrac{1}{2}+k+\mu, b_2, b_2; 1+2\mu, c_2, c_3; -\frac{p}{\alpha}, \frac{y}{\alpha}, \frac{z}{\alpha}\right)$$

valid for $R(\lambda \pm \mu) > 0$, $R(l-\mu-k+\frac{1}{2}) > 0$, $p, \alpha, y, z > 0$,

$$p+(\sqrt{y}+\sqrt{z})^2 < \alpha. \qquad \ldots.(25)$$

The following results can be derived by using the results (9) to (12) and Erdélyi [1, p. 222-223]

$$\int_0^\infty t^{l-\mu-k-\frac{1}{2}} (t+p+\alpha)^{\mu-l-k}\ {}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu; \tfrac{1}{2}+l-k-\mu; -\frac{t}{p}\right)$$

$$F_1\left(l+\lambda-\mu, b_2\, b_3; c_2; \frac{y}{t+p+\alpha}, \frac{z}{t+p+\alpha}\right) dt =$$

$$\frac{\Gamma(\frac{1}{2}+l-\mu-k)}{\Gamma(\lambda+l-\mu)} p^{\frac{1}{2}-k} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)\ \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k)\ \alpha^{\lambda+\mu}} p^{\mu}$$

$$F_G\left(\lambda+\mu, \lambda+\mu, \lambda+\mu, \tfrac{1}{2}+k+\mu, b_2, b_3; 1+2\mu, c_2, c_2; -\frac{p}{\alpha}, \frac{y}{\alpha}, \frac{z}{\alpha}\right)$$

valid for $R(\frac{1}{2}+l-\mu-k) > 0$, $R(\lambda \pm \mu) > 0$, $p, \alpha, y, z > 0$,

$$p+y < \alpha, p+z < \alpha. \qquad \ldots.(26)$$

$$\int_0^\infty t^{l-\mu-k-\frac{1}{2}} (t+p+\alpha)^{\mu-l-\lambda}\ {}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu; \tfrac{1}{2}+l-k-\mu; -\frac{t}{p}\right)$$

$$F_2\left(b_1, a_1, l+\lambda-\mu; c_1, c_2; x, \frac{z}{t+p+\alpha}\right) dt =$$

$$\frac{\Gamma(\frac{1}{2}+l-\mu-k)}{\Gamma(\lambda+l-\mu)} p^{\frac{1}{2}-k} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)\quad \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k)\ \alpha^{\lambda+\mu}} p^{\mu}$$

$$F_K\left(a_1, \lambda+\mu, \lambda+\mu, b_1, \tfrac{1}{2}+k+\mu, b_1; c_1, 1+2\mu, c_3; x, -\frac{p}{\alpha}, \frac{z}{\alpha}\right)$$

valid for $R(\frac{1}{2}+l-\mu-k) > 0$, $R(\lambda \pm \mu) > 0$, $x, \alpha, p, z > 0$,

$$z < (1-x)(\alpha-p). \qquad \ldots.(27)$$

$$\int_0^\infty t^{l-\mu-k-\frac{1}{2}} (t+p+\alpha)^{\mu-l-\lambda}\ {}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu; \tfrac{1}{2}+l-k-\mu; -\frac{t}{p}\right)$$

$$F_1\left(a_1, l+\lambda-\mu, b_2; c_2; \frac{z}{t+p+\alpha}, y\right) dt$$

$$= \frac{\Gamma(\frac{1}{2}+l-\mu-k)}{\Gamma(\lambda+l-\mu)} p^{\frac{1}{2}-k} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)\quad \Gamma(\lambda+\mu)}{\Gamma(\frac{1}{2}-\mu-k)\ \alpha^{\lambda+\mu}} p^{\mu}$$

$$F_M\left(\tfrac{1}{2}+k+\mu, a_2, a_2, \lambda+\mu, b_2, \lambda+\mu ; 1+2\mu, c_2, c_2 ; -\frac{p}{\alpha}, y, \frac{z}{\alpha}\right)$$

valid for $R(\frac{1}{2}+l-\mu-k)>0$, $R(\lambda\pm\mu)>0$, $p, \alpha, y, z>0$,
$p+z<\alpha, y<1$. .... (28)

$$\int_0^\infty t^{l-\mu-k-\frac{1}{2}}(t+p+\alpha)^{\mu-l-\lambda}\,{}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu; \tfrac{1}{2}+l-k-\mu; -\frac{t}{p}\right)$$

$$F_3\left(a_2, a_3, b_2, \lambda+l-\mu ; c_2 ; y, \frac{z}{t+p+\alpha}\right) dt$$

$$=\frac{\Gamma(\frac{1}{2}+l-\mu-k)}{\Gamma(\lambda+l-\mu)}\, p^{\frac{1}{2}-k} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} \frac{\Gamma(\lambda+\mu)}{\alpha^{\lambda+\mu}} p^{\mu}$$

$$F_N\left(\tfrac{1}{2}+k+\mu, a_2, a_3, \lambda+\mu, b_2, \lambda+\mu, 1+2\mu, c_2, c_2 ; -\frac{p}{\alpha}, y, \frac{z}{\alpha}\right)$$

valid for $R(\frac{1}{2}+l-\mu-k)>0$, $R(\lambda\pm\mu)>0$, $p, \alpha, y, z>0$,
$y(\alpha-p)+z(1-y)<0$. .... (29)

In particular, let $y\to 0$, then $F_N\to F_2$, from (29), we obtain the following interesting integral

$$\int_0^\infty t^{l-\mu-k-\frac{1}{2}}(t+p+\alpha)^{\mu-l-\lambda}\;{}_2F_1\left(\tfrac{1}{2}-k-\mu, \tfrac{1}{2}-k+\mu ; \tfrac{1}{2}+l-k-\mu ; -\frac{t}{p}\right)$$

$${}_2F_1\left(a_3, \lambda+l-\mu ; c_2 ; \frac{z}{t+p+\alpha}\right) dt$$

$$=\frac{\Gamma(\frac{1}{2}+l-\mu-k)}{\Gamma(\lambda+l-\mu)}\, p^{\frac{1}{2}-k} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} \frac{\Gamma(\lambda+\mu)}{\alpha^{\lambda+\mu}} p^{\mu}$$

$$F_2\left(\lambda+\mu, \tfrac{1}{2}+k+\mu, a_3 ; 1+2\mu, c_2 ; -\frac{p}{\alpha}, \frac{z}{\alpha}\right)$$

valid for $R(\frac{1}{2}+l-\mu-k)>0$, $R(\lambda\pm\mu)>0, p, \alpha, z>0, p+z<\alpha$. .... (30)

REFERENCES

1. Erdélyi, A. *et al.* Tables of integral transforms. Vol. I (McGraw-Hill; New York, (1954).
2. Rathie, C. B. "Some results involving Hypergeometric and E-functions". *Proc. Glasg. Math Ass.* III *July*, 132–138, (1955).
3. Saxena, R. K. "Integrals involving Legendre functions". *Math. Annalen.* **147** : 154 157, (1962).
4. Saran, S. Hypergeometric functions of three variables. *Ganita*, 5 : 77 91, (1954).
5. *Ibid.* Integral representations of Laplace type for certain hypergeometric functions of three variables. *Riv. Mat. Univ. Parma*, **8** : 133-143 (1957).
6. Varma, R. S. On a generalization of Laplace integral. *Proc. Nat Acad. Sci. India*, Sec. A, **20** ; 209-216, (1951).

CERTAIN CONVERGENCE THEOREMS AND ASYMPTOTIC PROPERTIES OF A GENERALIZATION OF LOMMEL AND MAITLAND TRANSFORM

By

RAM SHANKAR PATHAK

Department of Mathematics, Banaras Hindu University, Varanasi–5

[Received on 23rd April, 1965]

ABSTRACT

In previous papers, I have studied the properties of the generalized transform

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy)\, g(y)\, dy,$$

which reduces to Lommel transform (Hardy 1925) for $\mu = 1$ and to the generalized Hankel transform (Agarwal 1950) for $\lambda = 0$.

In the present paper, I give herewith certain convergence and order theorems for this generalization.

1. Recently I introduced a generalization of the wellknown Lommel transform (Hardy 1925) *viz.*,

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} F_\nu(xy)\, g(y)\, dy, \tag{1·1}$$

where
$$F_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r+2\lambda}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+r)}$$
$$= \frac{2^{2-\nu-2\lambda}}{\Gamma(\lambda)\,\Gamma(\nu+\lambda)}\, s_{\nu+2\lambda-1,\,\nu}(x)$$

and the generalized Hankel transform (Agarwal 1950), which we call as the Maitland transform, *viz.*,

$$f(x) = (\tfrac{1}{2})^{\nu} \int_0^\infty (xy)^{\nu+\frac{1}{2}} J^{\mu}_{\nu}\left(\frac{x^2y^2}{4}\right) g(y)\, dy,\ (\mu > 0), \tag{1·2}$$

where
$$J^{\mu}_{\nu}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{\lfloor r\ \Gamma(1+\nu+\mu r)},\quad (\mu > 0),$$

is the Bessel-Maitland function ; in the form :

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy)\, g(y)\, dy, \tag{1·3}$$

where
$$J^{\mu}_{\nu,\lambda}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r+2\lambda}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+\mu r)},\quad (\mu>0).$$

It may be noted that (1·3) reduces to (1·1) for $\mu=1$ and to (1·2) for $\lambda=0$.

In the present paper we shall discuss the convergence properties of the integral

$$\int_0^\infty (xy)^{\frac{1}{2}}\; J^{\mu}_{\nu,\lambda}(xy)\; d\alpha(y) \tag{1·4}$$

instead of (1·3), where $\alpha(y)$ is a function of the real variable y in the interval $0 \leqslant y < \infty$ and is of bounded variation in $0 \leqslant y \leqslant R$ for every positive R.

When the integral (1·4) converges, it defines a funtion of x which we shall call the $J^{\mu}_{\nu,\lambda}$ – Stieltjes transform of $\alpha(y)$ and shall denote it by $f(x)$. If

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xy)\, g(y)\, dy,$$

we shall refer to $f(x)$ as the $J^{\mu}_{\nu,\lambda}$ – transform of $g(y)$.

2. Theorem 1. If

$$\underset{0 \leqslant u < \infty}{u.\,b.} \left| \int_0^u (xy)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) \right| = M < \infty, \tag{2·1·1}$$

then the integral

$$\int_0^\infty (xy)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) \tag{2·1·2}$$

converges for $x > x_0$, $0 < \mu < 1$ with $R(\nu+2\lambda) < 2$ when $R(\lambda)$, $R(\nu+\lambda-\mu+1) \neq 0, -1, -2, \ldots$; and is equal to

$$\left(\frac{x}{x_0}\right)^{\frac{1}{2}} \int_0^\infty \psi(xy)\, \beta(y)\, dy, \tag{2·1·3}$$

where

$$\psi(xy) = \frac{x_0 J^{\mu}_{\nu+1,\,\lambda-1}(x_0 y)\, J^{\mu}_{\nu,\lambda}(xy) - x J^{\mu}_{\nu+1,\,\lambda-1}(xy)\, J^{\mu}_{\nu,\lambda}(x_0 y)}{\left[J^{\mu}_{\nu,\lambda}(x_0 y)\right]^2} \tag{2·1·4}$$

and

$$\beta(u) = \int_0^u (x_0 y)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(x_0 y)\, d\alpha(y). \tag{2·1·5}$$

Proof: If $\beta(u)$ is defined by (2·1·5), then (Widder, 1963, p. 12)

$$\left(\frac{x_0}{x}\right)^{\frac{1}{2}} \int_0^R (xy)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) = \int_0^R \frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)}\, d\beta(y),$$

$$= \left[\frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \beta(y) \right]_0^R - \int_0^R \frac{d}{dy} \left\{ \frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \right\} \beta(y)\, dy.$$

Now, $\left[\dfrac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \right] \beta(y)$ vanishes at $y = 0$ for $J^{\mu}_{\nu,\lambda}(y) = 0\,(y^{\nu+2\lambda})$,

when $y \to 0$ and $\beta(y)$ vanishes at $y = 0$ by virtue of (2·1·5).

Since (Wright 1935)

$$J^{\mu}_{\nu,\lambda}(x) \sim \frac{x^{\nu+2\lambda-2}}{\Gamma(\lambda)\ \Gamma(\nu+\lambda-\mu+1)} + x^{\nu+2\lambda-2k(\nu+2\lambda+\frac{1}{2})} \exp\left\{ \left(\mu \frac{x^2}{4} \right)^k \frac{\cos \pi k}{\mu k} \right\}$$

for large values x, $0 < \mu \leqslant 1$ and $k = \dfrac{1}{1+\mu}$.

So that

$$\lim_{R\to\infty} \left[\frac{J^{\mu}_{\nu,\lambda}(xR)}{J^{\mu}_{\nu,\lambda}(x_0 R)} \right] \beta(R) = 0, \text{ where } x > x_0 \text{ and } 0 < \mu < 1$$

with $R(\nu+2\lambda) < 2$ when $R(\lambda)$, $R(\nu+\lambda-\mu+1) \neq 0, -1, -2 \ldots$.

Now

$$\left| \int_0^R \frac{d}{dy} \left\{ \frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \right\} \beta(y)\, dy \right| \leqslant M \int_0^R \frac{d}{dy} \left\{ \frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \right\} dy = M \left[\frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \right]_0^R$$

From the above discussions, it is clear that, under the conditions stated, the right hand side of this inequality is finite, however large R may be. Therefore, by virtue of the relation (2·1·3), we have

$$\int_0^{\infty} (xy)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) = -\left(\frac{x}{x_0} \right)^{\frac{1}{2}} \int_0^{\infty} \frac{d}{dy} \left\{ \frac{J^{\mu}_{\nu,\lambda}(xy)}{J^{\mu}_{\nu,\lambda}(x_0 y)} \right\} \beta(y)\, dy$$

$$= -\left(\frac{x}{x_0} \right)^{\frac{1}{2}} \int_0^{\infty} \psi(xy)\, \beta(y)\, dy,$$

where $$\psi(xy) = \frac{-x J^{\mu}_{\nu+1,\lambda-1}(xy) J^{\mu}_{\nu,\lambda}(x_0 y) + x_0 J^{\mu}_{\nu+1,\lambda-1}(x_0 y) J^{\mu}_{\nu,\lambda}(xy)}{\left[J^{\mu}_{\nu,\lambda}(x_0 y) \right]^2},$$

on using the formula (Pathak, 1963) :

$$x J'^{\mu}_{\nu,\lambda}(x) = \nu J^{\mu}_{\nu,\lambda}(x) + x J^{\mu}_{\nu+1,\lambda-1}(x). \tag{A}$$

Hence the theorem.

Corollary. If the integral (2·1·2) converges for $x = x_0$, then, it converges for all $x > x_0$.

For, if (2·1·2) converges for $x = x_0$, then (2·1·1) holds and hence (2·1·2) converges for all $x > x_0$.

Theorem 2. If the integral

$$\int_0^\infty (xy)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) \tag{2·2·1}$$

converges, then

$$\alpha(y) = o\left(y^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)\right)^{-1} \tag{2·2·2}$$

for every positive x and $0 \leq \mu < 1$ with $R(\nu + 2\lambda) \leq 3/2$ when $R(\lambda)$, $R(\nu+\lambda - \mu + 1) \neq 0, -1, -2, \ldots$

Proof : Let

$$\beta(t) = \int_0^t (xy)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y),\ 0 \leq t \leq \infty\ ; \tag{2·2·3}$$

then (Widder, p. 12)

$$\int_0^t d\alpha\,(y) = \int_0^t \frac{d\beta(y)}{(xy)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)}\ .$$

Or,

$$\alpha(t) - \alpha(0) = \frac{\beta(t)}{(xt)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xt)} - \int_0^t \beta\,(y) \quad \frac{d}{dy}\left[(xy)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)\right]^{-1} dy.$$

Since

$$\left[\frac{\beta(y)}{(xy)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)}\right] \to 0 \text{ as } y \text{ tends to zero by virtue of (2·2·3)}$$

and the property $\qquad J^{\mu}_{\nu,\lambda}(y) = 0\ (y^{\nu+2\lambda}\ )$, when $y \to 0$.

Also, by hypothesis, $\beta(\infty)$ exists. Therefore

$$\lim_{t\to\infty}\left[\alpha(t) - \alpha(0)\right](xt)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xt)$$

$$= \lim_{t\to\infty} (xt)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xt) \int_0^t \left[\beta(\infty) - \beta(y)\right] \frac{d}{dy}\left[(xy)^{\frac{1}{2}}\, J^{\mu}_{\nu,\lambda}(xy)\right]^{-1} dy,$$

where $0 \leq \mu \leq 1$ with $R(\nu + 2\lambda) < \frac{3}{2}$ in case $R(\lambda)$, $R(\nu + \lambda - \mu + 1) \neq 0, -1, -2, \ldots$

Since the right hand side vanishes under conditions of the theorem, we have

$$\alpha(t) - \alpha\,(0) = o\left[(xt)^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)\right]^{-1}, \quad t \to \infty$$

or

$$\alpha(y) = o\left[y^{\frac{1}{2}} \quad J^{\mu}_{\nu,\lambda}(xy)\right]^{-1},$$

for every positive x and $0 < \mu < 1$, with $R(\nu + 2\lambda) < \frac{3}{2}$ when $R(\nu + \lambda - \mu + 1) \neq 0, -1, -2, \ldots$

Hence the theorem.

Theorem 3. If

$$\alpha(y) = o\left[y^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right]^{-1}, \tag{2·3·1}$$

then the integral

$$\int_0^{\infty} (xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) \tag{2·3·2}$$

converges for all x, $R(\nu + 2\lambda) > -\frac{1}{2}$ and $\mu > 0$ with additional conditions $R(\nu+2\lambda) \geqslant \frac{3}{2}$ and $R(\lambda)$ $R(\nu + \lambda - \mu + 1) \neq 0, -1, -2, \ldots$, when $0 < \mu < 1$.

Proof :

Since $\alpha(y)$ is of bounded variation in every finite interval, it follows from (2·3·1) that there exists a constant M such that

$$\left| \alpha(y) \right| \leqslant M \left[y^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right]^{-1},$$

where $0 \leqslant y < \infty$. Hence

$$\int_0^R \frac{d}{dy}\left[(xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right] \alpha(y)\, dy \lll M \left[R^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xR) \right]^{-1} \int_0^R \frac{d}{dy}\left[(xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right] dy$$

$$= M \left[(xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right]_0^R \left[R^{1/2} J^{\mu}_{\nu,\lambda}(xR) \right]^{-1}.$$

Therefore the integral on the left hand side of the inequality converges for $R(\nu + 2\lambda) > -\frac{1}{2}$ and $\mu > 0$ with additional conditions $R(\nu + 2\lambda) \geqslant \frac{3}{2}$ and $R(\lambda)$, $R(\nu + \lambda - \mu + 1) \neq 0, -1, -2, \ldots$, when $0 < \mu < 1$.

Now

$$\int_0^R (xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy)\, d\alpha(y) = (xR)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xR)\, \alpha(R) - \int_0^R \frac{d}{dy}\left[(xy)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right] \alpha(y)\, dy.$$

But

$$(xR)^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xR)\, \alpha(R) \to 0 \text{ as } R \to \infty.$$

Hence (2·3·2) converges under the conditions of the theorem.

Corollary. If $\alpha(\infty)$ exists and if

$$\alpha(y) - \alpha(\infty) = o\left[y^{\frac{1}{2}} J^{\mu}_{\nu,\lambda}(xy) \right]^{-1},$$

The symbol $\lll$ means that the integral on the left is dominated by the integral on the right, or that the integral of the first is in absolute value not greater than that of the second over the whole range of integration.

then the integral (2·3·2) converges for all x, when $R(\nu+2\lambda) > -\frac{1}{2}$ and $\mu > 0$ with additional conditions $R(\nu+2\lambda) > 3/_2$ and $R(\lambda)$, $R(\nu+\lambda-\mu+1) \neq 0, -1, -2, \ldots$, in case $0 < \mu < 1$.

Theorem 4. If the integral

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \, d\alpha(y)$$

converges for all positive x, $0 < \mu < 1$ and $-\frac{1}{2} < R(\nu+2\lambda)$ with $R(\nu+2\lambda) < 3/_2$ in case $R(\lambda)$, $R(\nu+\lambda-\mu+1) \neq 0, -1, -2, \ldots$,

then

$$\int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \, d\alpha(y)$$

$$= -(\nu+\tfrac{1}{2}) \int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \frac{\alpha(y)}{y} \, dy - x \int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu+1,\,\lambda-1}(xy) \, \alpha(y) \, dy.$$

Proof: We have

$$\int_0^R (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \, d\alpha(y)$$

$$= (xR)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xR) \; \alpha(R) - \int_0^R \frac{d}{dy}\Big[(xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \Big] \alpha(y) \, dy,$$

provided $R(\nu+2\lambda) > -\frac{1}{2}$.

Now, by theorem 3, we have

$$(xR)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xR) \; \alpha(R) \to 0 \text{ as } R \to \infty.$$

Also, on making use of (A), we have

$$\frac{d}{dy}\Big[(xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \Big] = (\nu+\tfrac{1}{2}) \left(\frac{x}{y}\right)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) + x(xy)^{1/2} J^{\mu}_{\nu+1,\,\lambda-1}(xy).$$

Thus, we get

$$\int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \, d\alpha(y) = -(\nu+\tfrac{1}{2}) \int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu,\lambda}(xy) \frac{\alpha(y)}{y} \, dy$$

$$- x \int_0^\infty (xy)^{\frac{1}{2}} \; J^{\mu}_{\nu+1,\,\lambda-1}(xy) \, \alpha(y) \, dy,$$

valid under the conditions of the theorem.

My best thanks are due to Professor Dr. Brij Mohan of the Banaras Hindu University for his kind help in the preparation of this paper.

REFERENCES

1. Agarwal, R. P. *Annals de la Soc. Scient. Bruxelles*, **64** : 161–168, (1950).
2. Hardy, G. H. *Proc Lond. Math. Soc.*, **53** : IX, (1925).
3. Pathak, R. S. In the Press (1963),
4. Widder, D. V. The Laplace Transform, *Princeton University Press*, (1963).
5. Wright, E. M. *J. London Math. Soc.*, **10** : 287–293, (1935).

ON THE MINIMUM CENTRAL PRESSURE OF A STAR

By

SHAMBHUNATH SRIVASTAVA

Department of Mathematics, K. N. Government College, Gyanpur, Varanasi

[Received on 1st January, 1966]

ABSTRACT

It has generally been believed that the minimum central pressure in a star of mass M and radius R, subject only to the condition that density does not increase inward is $3GM^2/8\pi R^4$ (Eddington, A. S., 1926). In this note author intends to show that the minimum central pressure is independent of the mass and radius of the star.

Eddington showed that any transfer of matter inwards increases the central pressure. We can, therefore, go on reducing the central pressure by transferring the matter outwards untill we arrive at uniform density. Then the central pressure shall obviously be the minimum. Eddington further assumed the relevancy of Emden's equation for $n = 0$ and considered it as a sphere of uniform density. He accordingly pointed out that the pressure corresponding to $n = 0$, given by

$$P = \frac{3}{8\pi} \frac{GM^2}{R^4}, \tag{1}$$

is the required minimum central pressure.

From $P = K\rho^{1+1/n}$, it is clear that when $n = 0$ and $\rho > 1$, then P is infinitely large. Indeed, an infinitely large pressure cannot be a minimum. If $n = 0$ and $\rho < 1$; then the pressure reduces to zero. The central pressure cannot be zero, for the central pressure howsoever small it may be, is the maximum in the configuration. Thus we see that a finite minimum central pressure cannot correspond to $n = 0$. Hence the minimum central pressure given by (1) should be reviewed.

Actually, as Aller pointed out (1954) Emden's equation can be integrated only for $n = 5$ and $n = 1$. Probably when he made this statement, he was aware of irrelevancy of $n = 0$.

We may however see it as follows :

Equation governing the configuration is (Chandrasekhar S., 1939)

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho. \tag{2}$$

Substitutions

$$\rho = \lambda\theta^n \quad ; \quad P = K\rho^{1+1/n} = K\lambda^{1+1/n}\,\theta^{n+1}, \tag{3}$$

transform (2) into

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \tag{4}$$

where

$$r = \alpha\xi \quad ; \quad \alpha = \left[\frac{(n+1)\,K}{4\,\pi\,G}\,\lambda^{1/n-1}\right]^{\frac{1}{2}}. \tag{5}$$

Equation (2) contains two dependent variables : P and ρ. $P = K\rho^{1+1/n}$ therefore is sufficient to eliminate one dependent variable from (2). Hence the only object of $\rho = \lambda\theta^n$ in (3) is to put the resulting equation in a proper form : equation (4). But for $n = 0$, $\rho = \lambda\theta^n$ makes ρ a constant ; $\rho = \lambda\theta^n$ therefore cannot be used as a transformer. This proves the irrelevancy of (4) for $n = 0$. It is further verified from the fact that α in (5), for $n = 0$, is either infinite or zero depending on weather $\lambda > 1$ or $\lambda < 1$; and hence the transformations in (5) cannot be relevant.

From the pressure density relation $P = K\rho^{1+1/n}$, it is clear that $n = 0$ and $n = -1$ are two values of n which can be relevant in the parts of the configuration in which pressure and density are constant without vanishing simultaneously. This shows that both $n = 0$ and $n = -1$ can be regarded as a sphere of uniform density. We have already seen that $n = 0$ cannot correspond to a finite minimum central pressure. For $n = -1$, the pressure is given by

$$P = \frac{k}{\mu H}\,\theta\gamma, \tag{6}$$

where k is the Boltzman constant, μ the molecular weight, H the mass of the proton and $\theta\gamma$, is the polytropic temperature. Pressure given by (6) is the required minimum central pressure. This is a constant quantity for a star and is independent of the mass and the radius of the star.

Author is extremely grateful to Dr. Brij Basi Lal, Professor and Head of the Department of Mathematics, K. N. Government College, Gyanpur (Varanasi) for his helpful discussions. Author is also thankful to the University Grants Commission for awarding the Junior Fellowship.

REFERENCES

1. Eddington, A. S. International Constitution of Star, p. 91, § 64, (1926).
2. Lawrence H. Aller. Astrophysics. *Nuclear Transformations, Stellar Interiors and Nebulae*, p. 55, (1954).
3. Chandrasekhar, S. An Introduction to the Study of Stellar Structure. (*Chicago : University of Chicago Press*), p. 87, eq. 6, (1939).

SOME THEOREMS IN OPERATIONAL CALCULUS

By

H. B. MALOO

Department of Mathematics, M. R. Engineering College, Jaipur

[Received on 23rd April, 1965]

1. The object of the present note is to evaluate some infinite integrals by making use of some theorems in Operational Calculus. The Laplace transform and Hankel transform of the function $f(t)$ are given by the integral equations

$$\phi(p) = p\int_0^\infty e^{-pt} f(t)\, dt \tag{1·1}$$

$$\psi(p) = p\int_0^\infty (pt)^{\frac{1}{2}} J_\nu(pt) f(t)\, dt. \tag{1·2}$$

We shall denote (1·1) and (1·2) symbolically as

$$\phi(p) \doteqdot f(t) \text{ and } \psi(p) \underset{\nu}{\overset{J}{\doteqdot}} f(t) \qquad \text{respectively.}$$

2. Theorem 1. If

$$\phi(p) \doteqdot f(t)$$

and

$$\psi(p) \underset{\nu}{\overset{J}{\doteqdot}} t^{-\frac{3}{2}} e^{-ct} J_\nu(bt)\, \phi(t).$$

then

$$\psi(p) = \frac{p}{\pi^{\frac{1}{2}}\, b^{\frac{1}{2}}} \int_0^\infty Q_{\nu-\frac{1}{2}}\left[\frac{(t+c)^2 + p^2 + b^2}{2pb}\right] f(t)\, dt, \tag{2·1}$$

provided that the integral is convergent and $R(c) > 0, p > 0, b > 0.$

Proof : We know that

$$f(t) \doteqdot \phi(p)$$

and [2, p. 183 (14)]

$$e^{-ct} J_\nu(at) J_\nu(bt) \doteqdot \frac{p}{\pi^{\frac{1}{2}}\, a^{\frac{1}{2}}\, b^{\frac{1}{2}}} Q_{\nu-\frac{1}{2}}\left[\frac{(p+c)^2 + a^2 + b^2}{2ab}\right]$$

for $R(p+c) > 0,\ R(\nu) > -\frac{1}{2}$

Using these relations in Parseval Goldstein theorem of Operational Calculus that if $\phi(p) \doteqdot f(t)$ and $\psi(p) \doteqdot g(t)$,

then

$$\int_0^\infty \phi(t)\, g(t)\, t^{-1}\, dt = \int_0^\infty \psi(t) f(t)\, t^{-1}\, dt, \tag{2·2}$$

we get

$$\int_0^\infty t^{-1} J_\nu(at) J_\nu(bt) e^{-ct} \phi(t)\, dt$$

$$= \frac{1}{\pi^{\frac{1}{2}} a^{\frac{1}{2}} b^{\frac{1}{2}}} \int_0^\infty \mathfrak{Q}_{\nu-\frac{1}{2}} \left[\frac{(t+c)^2+a^2+b^2}{2ab}\right] f(t)\, dt.$$

Replacing a by p and using (1·2) we get the result.

(2·2) is also true for Hankel transform.

Example. Starting from [2, p. 278 (23)]

$$\phi(p) = t^{\mu} e^{cp} K_\rho(cp)$$

$$\doteqdot 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} c^{-\frac{1}{2}} (t+2ct)^{-\frac{1}{2}\mu+\frac{1}{4}} P^{\mu-\frac{1}{2}}_{\rho-\frac{1}{2}}\left(1+\frac{t}{c}\right)$$

$$= f(t), \text{ for } |\arg c| < \pi,\ R(\mu) < \tfrac{3}{2};$$

we then have [1 p. 38]

$$t^{-\frac{3}{2}} e^{-ct} J_\nu(bt)\phi(t) = t^{\mu-\frac{3}{2}} J_\nu(bt) K_\rho(ct)$$

$$\frac{J}{\nu}\ \frac{2^{\mu-2}\, p^{\nu+\frac{3}{2}}\, b^\nu c^{-\mu-2\nu}\, \Gamma\left(\frac{\mu+2\nu-\rho}{2}\right)\Gamma\left(\frac{\mu+2\nu+\rho}{2}\right)}{\Gamma(1+\nu)\ \Gamma(1+\nu)}$$

$$\times F_4\left(\frac{\mu+2\nu-\rho}{2}, \frac{\mu+2\nu+\rho}{2}; 1+\nu, 1+\nu; -\frac{p^2}{c^2}, -\frac{b^2}{c^2}\right)$$

$$= \psi(p) \text{ for } R(\mu+2\nu\pm\rho)>0,\ R(c)>0,\ p>0,\ \mu>0.$$

Using the theorem we get

$$\int_0^\infty (t^2+2ct)^{-\frac{1}{2}\mu+\frac{1}{4}} P^{\mu-\frac{1}{2}}_{\rho-\frac{1}{2}}\left(1+\frac{t}{c}\right)\mathfrak{Q}_{\nu-\frac{1}{2}}\left[\frac{(t+c)^2+p^2+b^2}{2pb}\right] dt$$

$$= \frac{2^{\mu-\frac{3}{2}}(pb)^{\nu+\frac{1}{2}}\, c^{\frac{1}{2}-\mu-2\nu}\, \Gamma\left(\frac{\mu+2\nu-\rho}{2}\right)\Gamma\left(\frac{\mu+2\nu+\rho}{2}\right)}{\Gamma(1+\nu)\ \Gamma(1+\nu)}$$

$$\times F_4\left(\frac{\mu+2\nu-\rho}{2}, \frac{\mu+2\nu+\rho}{2}; 1+\nu, 1+\nu; -\frac{p^2}{c^2}, -\frac{b^2}{c^2}\right), \tag{2·3}$$

$$R(\mu+2\nu\pm\rho)>0,\ R(c)>0,\ p>0,\ b>0,\ R(\mu)<\tfrac{3}{2}.$$

Theorem II.

If

$$\psi(p) \frac{J}{\nu} f(t)$$

and

$$\phi(p) \doteqdot t^{-\frac{1}{2}} J_{2\nu}(2a\, t^{1/2}) f(t),$$

then

$$\phi(p) = p\int_0^\infty t^{-\frac{1}{2}}(p^2+t^2)^{-\frac{1}{2}}e^{-\left(\frac{a^2 p}{p^2+t^2}\right)}J_\nu\left(\frac{a^2 t}{p^2+t^2}\right)\psi(t)\,dt, \tag{2·4}$$

provided that the integral is convergent and $R(p)>0$ $R(\nu)>-\frac{1}{2}$.

Proof : Now

and [3, p. 58 (14)] $\qquad f(t) \overset{J}{\underset{\nu}{=}} \psi(p)$

$$t^{\frac{1}{2}}(\beta^2+t^2)^{-\frac{1}{2}}e^{-\left(\frac{a^2\beta}{\beta^2+t^2}\right)}J_\nu\left(\frac{a^2 t}{\beta^2+t^2}\right)$$

$$\overset{J}{\underset{\nu}{=}} p^{\frac{1}{2}}e^{-\beta p}J_{2\nu}(2a\,p^{1/2}),$$

where $R(\beta)>0$ $R(\nu)>-\frac{1}{2}$.

Using these relations in generalised Parseval-Goldstein theorem we get

$$\int_0^\infty t^{-\frac{1}{2}}(\beta^2+t^2)^{-\frac{1}{2}}e^{-\left(\frac{a^2\beta}{\beta^2+t^2}\right)}J_\nu\left(\frac{a^2 t}{\beta^2+t^2}\right)\psi(t)\,dt$$

$$=\int_0^\infty t^{-\frac{1}{2}}e^{-\beta t}J_{2\nu}(2a\,t^{\frac{1}{2}})\,f(t)\,dt.$$

Multiplying both sides by β, replacing β by p and using (1·1) we get (2·4)

Example I. Starting from [3 p. 29 (7)]

$$f(t) = t^{\mu-1/2}e^{-bt}$$

$$\overset{J}{\underset{\nu}{=}} \frac{p^{\nu+\frac{1}{2}}\,\Gamma(\mu+\nu+1)}{2^\nu\,b^{\mu+\nu+1}\,\Gamma(1+\nu)}\times {}_2F_1\left(\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2};1+\nu;-\frac{p^2}{b^2}\right)$$

$$=\psi(p),\quad R(b)>0,\ R(1+\mu+\nu)>0.$$

we then have [2 p. 185 (35)]

$$t^{-\frac{1}{2}}J_{2\nu}(2a\,t^{\frac{1}{2}})\,f(t) = t^{\mu-1}e^{-bt}J_{2\nu}(2a\,t^{\frac{1}{2}})$$

$$\doteq \frac{\Gamma(\mu+\nu)\,a^{2\nu}\,p}{\Gamma(2\nu+1)\,(p+b)^{\nu+\mu}}\times$$

$${}_1F_1\left(\mu+\nu;2\nu+1;-\frac{a^2}{p+b}\right).$$

$$=\phi(p),\qquad R(\mu+\nu)>0,\ R(b+p)>0.$$

Using the theorem we get

$$\int_0^\infty t^{\nu+1}(p^2+t^2)^{-\frac{1}{2}}e^{-\left(\frac{a^2 p}{p^2+t^2}\right)}J_\nu\left(\frac{a^2 t}{p^2+t^2}\right)$$

$$\times\ {}_2F_1\left(\frac{\mu+\nu+1}{2},\frac{\mu+\nu+2}{2};1+\nu;-\frac{t^2}{b^2}\right)dt$$

$$= \frac{\Gamma(\mu+\nu)\,\Gamma(1+\nu)\,2^{\nu}\,a^{2\nu}\,b^{\mu+\nu+1}}{\Gamma(2\nu+1)\,\Gamma(1+\mu+\nu)} \times {}_1F_1\left(\mu+\nu\,;\,2\nu+1\,;\,-\frac{a^2}{p+b}\right)$$

for $R(\mu+\nu)>0$, $R(\nu+1)>0$. $|\arg p| < \frac{\pi}{2}$, $|\arg b| < \frac{\pi}{2}$. (2·5)

Example II. Starting from [3 p. 58 (16)]

$$f(t) = t^{-\frac{1}{2}} J_{2\nu}(2b\,t^{\frac{1}{2}})$$

$$\frac{J}{\nu}\, p^{\frac{1}{2}} J_{\nu}\left(\frac{b^2}{p}\right)$$

$$= \psi(p), \textit{ for } R(\nu) > -\tfrac{1}{2},$$

we then have [3 p. 51 (24)]

$$t^{-\frac{1}{2}} J_{2\nu}(2a\,t^{\frac{1}{2}})\,f(t) = t^{-1} J_{2\nu}(2a\,t^{\frac{1}{2}})\,J_{2\nu}(2b\,t^{\frac{1}{2}})$$

$$\doteqdot 2p \sum_{m=0}^{\infty} \frac{\Gamma(m+2\nu-1)}{\lfloor m\, \Gamma(m+2\nu+1)} \left(-\frac{b^2}{p}\right)^m$$

$$\times {}_2F_1\left(-m, -2\nu-m\,;\,1+2\nu\,;\,\frac{a^2}{b^2}\right)$$

$$= \phi(p), \text{ for } R(p) > 0\ a > 0,\ b > 0,\ R(\nu) > 0$$

Using the theorem, we get

$$\int_0^{\infty} (p^2+t^2)^{-\frac{1}{2}}\, e^{-\left(\frac{a^2 p}{p^2+t^2}\right)} J_{\nu}\left(\frac{b^2}{t}\right) J_{\nu}\left(\frac{a^2\,t}{p^2+t^2}\right) dt$$

$$= 2 \sum_{m=0}^{\infty} \frac{\Gamma(m+2\nu-1)}{\lfloor m\, \Gamma(m+2\nu+1)} \left(-\frac{b^2}{p}\right)^m$$

$$\times {}_2F_1\left(-m, -2\nu-m\,;\,1+2\nu\,;\,\frac{a^2}{b^2}\right).$$

for $R(p) > 0$, $b > 0$, $a > 0$, $R(\nu) > 0$.

Theorem III

If $\psi(p) \doteqdot f(t)$

and

$$\phi(p) \doteqdot J_{\nu}(at)\,\psi\left(\frac{1}{t}\right),$$

then

$$\phi(p) = 2p \int_0^{\infty} J_{\nu}\left\{\sqrt{2t(\sqrt{p^2+a^2}-p)}\right\} K_{\nu}\left\{\sqrt{2t(\sqrt{p^2+a^2}+p)}\right\} f(t)\,dt, \quad (2·7)$$

provided that the integral is convergent.

Proof: Now

$$\psi(p) \doteqdot f(t)$$

and [**3**, p. 30 (16)]

$$2p J_\nu \left\{ \sqrt{2p(\sqrt{\beta^2 + a^2} - \beta)} \right\} K_\nu \left\{ \sqrt{2p(\sqrt{\beta^2 + a^2} + \beta)} \right\}$$

$$\doteqdot t^{-1} e^{-\frac{\beta}{t}} J_\nu\left(\frac{a}{t}\right)$$

$$R(\beta) > |Im\ a|, \ R(p) > 0.$$

The theorem follows on using these relations in (2·2), making a slight change in the variable and then replacing β by p.

Example I. Take [2 p. 274 (12)]

$$\psi(p) = p^{1-\lambda} I_\mu\left(\frac{b}{p}\right)$$

$$\doteqdot \frac{2^{-\mu} b^\mu t^{\lambda+\mu-1}}{\Gamma(1+\mu)\ \Gamma(\lambda+\mu)}\ {}_0F_3\left(; \mu+1, \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; \frac{b^2t^2}{16}\right)$$

$$= f(t), \qquad R(\lambda+\mu) > 0, R(p) > 0.$$

Then [5 p. 110 (15)].

$$J_\nu(at)\ \psi\left(\frac{1}{t}\right) = t^{\lambda-1} J_\nu(at)\, I_\mu(bt)$$

$$\doteqdot \frac{a^\nu b^\mu\ \Gamma(\lambda+\mu+\nu)}{2^{\nu+\mu} p^{\lambda+\mu+\nu-1}\ \Gamma(1+\nu)\ \Gamma(1+\mu)}$$

$$\times F_4\left(\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2}; 1+\nu, 1+\mu; -\frac{a^2}{p^2}, \frac{b^2}{p^2}\right)$$

$$= \phi(p), \qquad R(\lambda+\mu+\nu) > 0, R(p) > R(b) + |Im\ a|$$

Applying the theorem and replacing $\sqrt{p^2+a^2} - p$ by $2\alpha^2$ and $\sqrt{p^2+a^2} + p$ by $2\beta^2$, we get

$$\int_0^\infty t^{\lambda+\mu-1} J_\nu(2\alpha t^{\frac{1}{2}})\, K_\nu(2\beta t^{\frac{1}{2}})\ {}_0F_3\left(; \mu+1, \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}, \frac{b^2t^2}{16}\right) dt$$

$$= \frac{\Gamma(\lambda+\mu)\ \Gamma(\lambda+\mu+\nu)}{2\Gamma(1+\nu)(\beta^2-\alpha^2)^{\lambda+\mu+\nu}}$$

$$\times F_4\left(\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2}; 1+\nu, 1+\mu; \frac{-4\alpha^2\beta^2}{(\beta^2-\alpha^2)^2}, \frac{b^2}{(\beta^2-\alpha^2)^2}\right)$$

$$R(\lambda+\mu+\nu) > 0, R(\lambda+\mu) > 0, R(\beta - \sqrt{b}) > |Im\ \alpha| \qquad (2{\cdot}8)$$

From (2·8) it is easy to deduce that

$$\int_0^\infty t^{2\lambda + 2\mu - 1} K_\nu(\alpha t) K_\nu(\beta t)\ {}_0F_3\left(; \mu + 1, \frac{\lambda + \mu}{2}, \frac{\lambda + \mu + 1}{2}, \frac{b^2 t^4}{16}\right) dt$$

$$= \frac{\Gamma(\lambda + \mu)}{2^{3 - 2\lambda - 2\nu}} \sum_{\nu, -\nu} \frac{\Gamma(-\nu)\ \Gamma(\lambda + \mu + \nu)\ \alpha\beta)^\nu}{(\beta^2 + \alpha^2)^{\lambda + \mu + \nu}}$$

$$\times F_4\left(\frac{\lambda + \mu + \nu}{2}, \frac{\lambda + \mu + \nu + 1}{2}; 1 + \nu, 1 + \mu; \frac{4\alpha^2\beta^2}{(\beta^2 + \alpha^2)^2}, \frac{16 b^2}{(\beta^2 + \alpha^2)^2}\right),$$

for $R(\lambda + \mu \pm \nu) > 0$, $\quad R(\alpha + \beta) > R(2\sqrt{b})$. (2·9)

Example II. Now take [2, p. 274 (15)]

$$\psi(p) = J_\nu\left(\frac{2b^{\frac{1}{2}}}{p^{\frac{1}{2}}}\right)$$

$$\doteqdot \frac{b^\nu\ t^\nu}{\Gamma(1 + \nu)\ \Gamma(2\nu + 1)}\ {}_0F_2(\nu + 1, 2\nu + 1; bt)$$

$$= f(t), \qquad \text{for } R(1 + \nu) > 0, \qquad R(p) > 0.$$

Then [3, p. 58 (17)]

$$J_\nu(at)\ \psi\left(\frac{1}{t}\right) = J_\nu(at) J_{2\nu}(2b^{\frac{1}{2}} t^{\frac{1}{2}})$$

$$\doteqdot p(p^2 + a^2)^{-\frac{1}{2}} e^{-\left(\frac{bp}{p^2 + a^2}\right)} J_\nu\left(\frac{ab}{p^2 + a^2}\right)$$

$$= \phi(p), \qquad R(\nu) > -\tfrac{1}{2}, \qquad R(p) > 0.$$

Using the theorem, we get as before

$$\int_0^\infty t^\nu J_\nu(2\alpha t^{\frac{1}{2}}) K_\nu(2\beta t^{\frac{1}{2}})\ {}_0F_2(\nu + 1, 2\nu + 1; -bt)\ dt$$

$$= \frac{\Gamma(1 + \nu)\ \Gamma(2\nu + 1)}{2b^\nu\ (\alpha^2 + \beta^2)} e^{-\frac{b(\beta^2 - \alpha^2)}{(\alpha^2 + \beta^2)^2}} J_\nu\left(\frac{2b\ \alpha\beta}{(\beta^2 + \alpha^2)^2}\right)$$

for $R(\nu) > -\frac{1}{2}$, $R(\beta) > | Im\ \alpha |$, $b > 0$. (2·9)

3. We now use Parseval Goldstein formula to evaluate two more integrals.

(*i*) Take [6, p. 342]

$$\phi(p) = p\{p + a + b^2 + c^2)^2 - 4b^2 c^2\}^{-\frac{1}{2}(\rho + \frac{1}{2})}$$

$$Q_{\nu - \frac{1}{2}}^{\rho + \frac{1}{2}}\left(\frac{p + a + b^2 + c^2}{2bc}\right)$$

$$\doteqdot \pi^{\frac{1}{2}} (bc)^{\frac{1}{2}} t^\rho e^{-(a + b^2 + c^2)t} I_\nu(2b\ ct)$$

$$= f(t),$$

for $R[p + a + (b - c)^2] > 0$ $R(\rho + \nu + 1) > 0$

and [2, p. 249 (9)]

$$\psi(p) = \frac{\Gamma(\rho + 1 - 2k}{a^k} p^{k-\rho} e^{\frac{1}{2}ap} W_{k,\mu}(ap)$$

$$\doteqdot t^{\rho-2k} {}_2F_1\left(\tfrac{1}{2} - k + \mu, \tfrac{1}{2} - k - \mu ; \rho - 2k + 1 ; -\frac{t}{a}\right)$$

$$= g(t), \text{ for } R(\rho - 2k + 1) > 0 \; |\arg a| < \pi.$$

Using these relations in (2·2), evaluating the integral on the right with the help of a known result [4, p. 174 (2)], and replacing $\frac{1}{2} - k + \mu$ by α, $\frac{1}{2} - k - \mu$ by β, we get

$$\int_0^\infty t^{\rho+\alpha+\beta-1}\{(t + a + b^2 + c^2)^2 - 4b^2 c^2\}^{-\frac{1}{2}[\rho+\frac{1}{2})}$$

$$\times Q_{\nu-\frac{1}{2}}^{\rho+\frac{1}{2}}\left(\frac{t + a + b^2 + c^2}{2bc}\right) {}_2F_1\left(\alpha, \beta ; \alpha + \beta + \rho ; -\frac{t}{a}\right) dt$$

$$= \frac{\pi^{\frac{1}{2}} (bc)^{\frac{1}{2}+\nu} \Gamma(\alpha + \beta + \rho)\, \Gamma(1 - \alpha + \nu)\, \Gamma(1 - \beta + \nu)\, a^{\alpha+\beta-\nu-1}}{\Gamma(1 + \nu)\, \Gamma(1 + \nu)}$$

$$\times F_4\left(1 - \alpha + \nu, 1 - \beta + \nu ; 1 + \nu, 1 + \nu ; -\frac{b^2}{a}, -\frac{c^2}{a}\right), \tag{3·1}$$

for $R(1 - \alpha + \nu) > 0$, $R(1 - \beta + \nu)$, $R(\alpha + \beta + \rho) > 0$, $|\arg a| < \pi$
$|\arg\{a + (b \pm c)^2\}| < \pi$.

(*ii*) Now take [2, p. 220, (19)].

$$\phi(p) = \Gamma(2\lambda + 2\mu)\, p(p + \alpha + \beta)^{-2\lambda-2\mu} {}_2F_1\left(\frac{2\lambda + 2\mu + 1}{4}, \frac{2\lambda + 2\mu + 3}{4} ; 1 + \mu ; \frac{16b^2}{(p + \alpha + \beta)^4}\right)$$

$$\doteqdot t^{2\lambda+2\mu-1} e^{-(\alpha+\beta)t} {}_0F_3\left(; \mu + 1, \frac{\lambda + \mu}{2}, \frac{\lambda + \mu + 1}{2} ; \frac{b^2 t^4}{16}\right)$$

$$= f(t) \text{ for } R(\lambda + \mu) > 0, R(p + \alpha + \beta - 2\sqrt{b}) > 0$$

and [2, p. 212 (6)]

$$\psi(p) = \frac{2\alpha^{\frac{1}{2}} \beta^{\frac{1}{2}}}{\pi} p\, e^{(\alpha+\beta)p} K_\nu(\alpha p)\, K_\nu(\beta p)$$

$$\doteqdot P_{\nu-\frac{1}{2}}\left[\frac{(t + 2\alpha)(t + 2\beta)}{2\alpha\beta} - 1\right]$$

$$= g(t).$$

Using these relations in (2·2) and evaluating the integral on the right with the help of (2·9) we get

$$\int_0^\infty (t + \alpha + \beta)^{-2\lambda - 2\mu} \, P_{\nu - \frac{1}{2}} \left[\frac{(t + 2\alpha)(t + 2\beta)}{2\alpha\beta} - 1 \right]$$

$$\times \, {}_2F_1 \left(\frac{2\lambda + 2\mu + 1}{4}, \frac{2\lambda + 2\mu + 3}{4} ; 1 + \mu ; \frac{16 b^2}{(t + \alpha + \beta)^4} \right) dt$$

$$= \frac{1}{2 \pi^{\frac{1}{2}} \Gamma(\lambda + \mu + \frac{1}{2})} \sum_{\nu, -\nu} \frac{(\alpha \beta)^{\nu + \frac{1}{2}} \Gamma(-\nu) \, \Gamma(\lambda + \mu + \nu)}{(\beta^2 + \alpha^2)^{\lambda + \mu + \nu}}$$

$$\times \, F_4 \left(\frac{\lambda + \mu + \nu}{2}, \frac{\lambda + \mu + \nu + 1}{2} ; 1 + \nu, 1 + \mu ; \frac{4\alpha^2 \beta^2}{(\beta^2 + \alpha^2)^2}, \frac{16 b^2}{(\beta^2 + \alpha^2)^2} \right).$$

valid by analytic continuation, for $R(\lambda + \mu \pm \nu) > 0$, (3·2)

$$R(\alpha + \beta) > R(2\sqrt{b}).$$

REFERENCES

1. Bailey, W. N. Some infinite integrals involving Bessel functions. *Proc. London Math. Soc.* Series 2 Vol. 40 pp. 37–48 (1935).
2. Erdelyi, A. Tables of integral transforms Vol. I. McGraw Hill, New York (1954).
3. ——— Tables of integral transforms Vol. II. McGraw Hill, New York, (1954).
4. Saxena, R. K. Integrals involving Bessel functions and Whittaker functions. *Proc. Cambridge Phil. Soc.* 60 pp. 174–76 (1964).
5. Sharma, K. C. Theorems relating Hankel and Meijer's Bessel transforms. *Proc. Glasgow Math. Assoc.* Vol. 6 part. 2 pp. 107–12 (1963).
6. MacRobert, T. M. *Spherical Harmonics.*

COMPLEX COMPOUNDS OF BLUE PERCHROMIC ACID WITH AMINES AND HETEROCYCLIC BASES PART—I

By

SARJU PRASAD, A. V. PANDU RANG RAO *and* RAJ KUMAR SINGH

Chemical Laboratories, Banaras Hindu University, Varanasi

[Received on 7th October, 1964]

Complexes of blue perchromic acid with primary secondary, tertiary amines and heterocyclic bases in other medium have been prepared, their properties studied and structures discussed.

INTRODUCTION

The deep blue colour produced when H_2O_2 is added to a solution containing chromic acid, was discovered by Barreswil[1]. Moisson[2] in 1883 extracted the blue perchromic acid in ether and on evaporating the blue etheral solution at -20^0C obtained a deep indigo blue oily liquid to which he assigned the formula $CrO_3.H_2O_2$. Various structures for perchromic acid have been suggested[3-4]. Pyridine and piperidine complexes of blue perchromic acid have been studied by R. C. Rai[5].

As practically no work has been done on the formation of the complex compounds of blue perchromic acid with amines and heterocyclic bases the present investigation was undertaken with a view to study these complexes in ether medium.

EXPERIMENTAL

Preparation of blue perchromic acid.—Hydrogen peroxide solution cooled by ice was added to an ice cooled aqueous solution containing chromic acid. The blue perchromic acid formed was extracted with ether by a separating funnel.

Preparation of the complex.—An ethereal solution of organic compound was added to perchromic acid solution cooled by ice when a precipitate was obtained. It was filtered, washed with ether and dried in a vacuum desiccator.

Analysis.—Chromium was estimated by the following two methods.

(*i*) A weighed quantity of the complex was taken in a silica crucible and heated first slowly then strongly. The complex decomposed leaving Cr_2O_3 as the residue. It was weighed and the percentage of CrO_5 in the compound calculated.

(*ii*) The complex was dissolved in HCl, decomposed by evaporating to dryness two to three times, extracted with dil. HCl. The solution was filtered and Cr precipitated as hydroxide by NH_4OH. It was filtered, washed free of electrolytes, dried, ignited, weighed as Cr_2O_3 and the percentage of CrO_5 calculated.

In a few cases nitrogen was estimated by Kjeldahl's method and the percentage of the organic substance calculated. In the rest it was found by difference (*cf.* table 1 and 2).

TABLE 1

Complex compounds of blue perchromic acid with primary amines

S. N.	Organic compounds	Molecular formula	Pysical state	Percentage of CrO_5		Percentage of Nitrogen		Percentage of Organic Matter	
				Found	Calc.	Found	Calc.	Found	Calc.
1.	α-naphthylamine	$CrO_5.C_{10}H_7.NH_2$	Light pink powder	49·25	47·96	4·91	5·09	..	..
2.	β-naphthylamine	do	Brown mass	49·09	47·96	..	..	50·91	52·04
3.	*o*-Toluidine	$CrO_5.C_6H_4CH_3NH_2$	Black solid	56·17	55·22	5·63	5·85	..	..
4.	*m*-Toluidine	do	Dark Brown Powder	56·46	55·22	..	..	43·54	44·78
5.	*p*-Toluidine	do	Snuff coloured powder	56·10	55·22	..	..	43·90	44·78
6.	Aniline	$CrO_5C_6H_5NH_2$	Dark brown solid	59·91	58·66	6·01	6·22	..	..
7.	*o*-phenetidine	$CrO_5C_6H_4OC_2H_5.NH_2$;	Dark violet	51·81	49·05	..	..	48·19	50·95
8.	*m*-phenetidine	do	Dark yellow	51·94	49·05	5·38	5·20	..	..
9.	*p*-phenetidine	do	Dark brown	51·45	49·05	..	..	48·55	50·95
10.	*o*-Anisidine	$CrO_5.C_6H_4OCH_3NH_2$	Shining black	52·55	51·76	..	..	47·45	48·24
11.	*m*-Anisidine	do	Snuff colour	52·93	51·76	..	..	47·07	48·24
12.	*p*-Anisidine	do	Dark brown	52·50	51·76	5·27	5·49	..	..
13.	Methylamine	$CrO_5.CH_3NH_2$	Dark red liquid	82·02	80·98	8·70	8·58	..	..
14.	*n*-propylamine	$CrO_5.C_3H_7NH_2$	Dark brown viscous liquid	70·85	69·12	7·16	7·33	..	..

TABLE 2

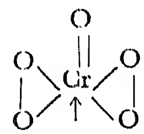
Complex compounds of blue perchromic acid with secondary, tertiary amines and heterocyclic bases

S. N.	Organic compounds	Molecular formula	Physical state	Percentage of CrO_5		Percentage of Nitrogen		Percentage of organic matter	
				Found	Calc.	Found	Calc.	Found	Calc.
1.	Dimethylaniline	$CrO_5.C_6H_5N(CH_3)_2$	Black solid	54·89	52·25	5·62	5·53	..	..
2.	Diethylamine	$CrO_5.NH(C_2H_5)_2$	Deep brown semi solid	63·10	64·40	6·39	6·82	..	..
3.	Triethylamine	$CrO_5.N(C_2H_5)_3$	Deep brown liquid becomes solid after keeping	57·16	56·66	6·15	6·00	..	..
4.	Nicotine	$CrO_5.C_{10}H_{14}N_2$	Black viscous liquid	46·23	44·74	9·28	9·52	..	..
5.	α–picoline	$CrO_5.C_6H_7N$	Deep brown solid	54·48	58·67	6·29	6·22	..	..
6.	β–picoline	do	Deep brown liquid becoming solid	56·82	58·67	..	..	43·18	41·33
7.	γ–picoline	do	Black semi solid	59·02	58·67	..	..	40·98	41·33
8.	Ethylaniline	$CrO_5.C_6H_5.NH.C_2H_5$	Black solid	48·80	52·25	5·64	5·53	..	..
9.	Quinoline	$CrO_5.C_9H_7N$	Brown powder	52·58	50·57	..	..	47·42	49·43
10.	Isoquinoline	do	Brown black solid	51·49	50·57	5·09	5·36	..	..
11.	Dimethylamine prophylamine	$CrO_5.C_3H_6NH_2N(CH_3)_2$	Deep brown liquid	55·91	56·41	11·68	11·97	..	..
12.	Pyrrolidine	$CrO_5.(CH_2)_4NH$	Black semi solid	64·18	65·04	6·72	6·89	..	..

General properties.—The compounds formed with primary monoamines are coloured powders and those with secondary andiary am tertines and heterocyclic bases are coloured semi solids or highly viscous liquids except those with dimethylaniline, ethylaniline and quinoline, which are coloured solids. They are insoluble or very sparingly soluble in organic solvents except those with dimethylaniline dimethylamine, triethylamine and α– and β– picolines which are soluble in water and alcohol. All the compounds decompose when treated with dilute acid or alkali solution.

DISCUSSION

It is observed that the compounds are formed more readily with monoamines than with secondary and tertiary amines which shows the weak coordinating power of –NH or N group. The compounds formed may be represented as



where the arrow indicates a lone pair of electrons from the nitrogen atom in the primary, secondary or tertiary amine or heterocyclic bases. The E. A. N. does not assume the inert gas configuration but the compounds are fairly stable in dry atmosphere which is probably due to the symmetrical structure which it assumes.

ACKNOWLEDGMENT

Author's sincere thanks are due to the authorities of the Banaras Hindu University for the facilities provided.

REFERENCES

1. Barreswil. *Compt. rend.*, **16** : 1085, 1848.
2. Moisson *Compt. rend.*. **97** : 96, 1883.
3. Weide. *Ber.*, **30** : 2178, 1897.
4. Schwarz and Gies. *Ber.*, **65B** : 871, 1932.
5. Rai, R. C. *J. I. C. S.* Vol. **34** : 68, 1957.

ACRYLATE AND CROTONATE COMPLEXES OF LANTHANUM

By

P. L. KACHROO *and* A. K. BHATTACHARYA

Chemical Laboratories, University of Saugar, Sagar, M. P.

[Received on 18th March, 1965]

ABSTRACT

1 : 2 Complexes of lanthanum with acrylic and crotonic acids have been reported. Conductance and spectrophotometric data have been employed as indicative properties. Job's method of continued variation has been used for the determination of the composition and the values for the dissociation constants of these complexes. The dissociation constants of the acrylate and the crotonate complexes were found to be $4{\cdot}26 \times 10^{-5}$ and $5{\cdot}52 \times 10^{-5}$ respectively at 26°C.

The unsaturated acids do not appear to have drawn much attention regarding their use as complexing agents. There is no record of any instance of complex formation of metals with unsaturated monocarboxylic acids except for the copper crotonate.[1] This communication deals with the systems $LaCl_3$ - acrylic acid and $LaCl_3$ - crotonic acid studied by the Job's method of continued variation.

EXPERIMENTAL

Lanthanum chloride, AnalaR B. D. H. acrylic and crotonic acids, and A. R. sodium carbonate were used. Doran's conductivity Bridge and Beckmann's spectrophotometer, Model DU, were employed for the experiments.

Stock solution of the known molarity of the sodium salts were prepared by direct weighing ; equivalent amounts of the acids and sodium carbonate were mixed, boiled to remove carbon dioxide and used as such after checking the pH's of the solutions.

Lanthanum chloride solution was standardised, both for lanthanum and chloride contents. Volumetric method suggested by Kolthoff and Elmquist[2] was used for the estimation of lanthanum and the chloride content was estimated gravimetrically as silver chloride.

COMPOSITION

Conductivity Method :— Several mixtures containing the metal salt and the ligand in different proportions, with the restriction that the total molarity and volume (20 ml.) remained constant, were prepared. Corresponding to each of the mixtures two blank solutions of lanthanum chloride and the ligand were prepared in such a way that their quantities in the mixture were the same as in the pure solutions. All the mixtures and the pure solutions were kept in an etectrically maintained thermostat at $26 \pm 0{\cdot}1$°C for about twelve hours to attain equilibrium and then their conductance was measured. Differences between conductance of the mixtures over the added value of conductance of the pure metal ion and the pure ligand were plotted against their composition (Fig. 1, 2). The maximum in each graph corresponds to the composition of the complex,

Spectrophotometric Method :—Mixtures containing the metal and the ligands in various molar proportions were prepared and the variation of optical density with wave length was observed. From these observations suitable wave lengths were chosen for study.

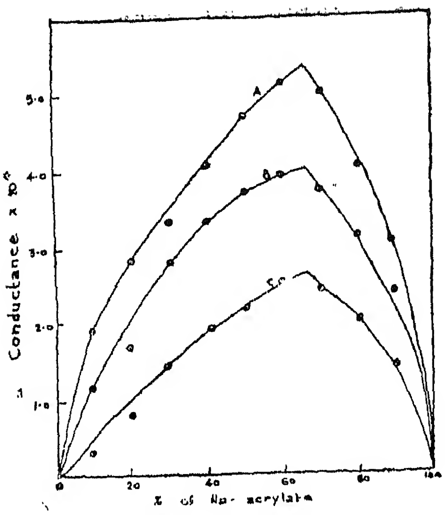


Fig. 1 Composition.
Lm. - Ligand
A. M/60 - M/60
B. M/80 - M/80
C. M/100 - M/100

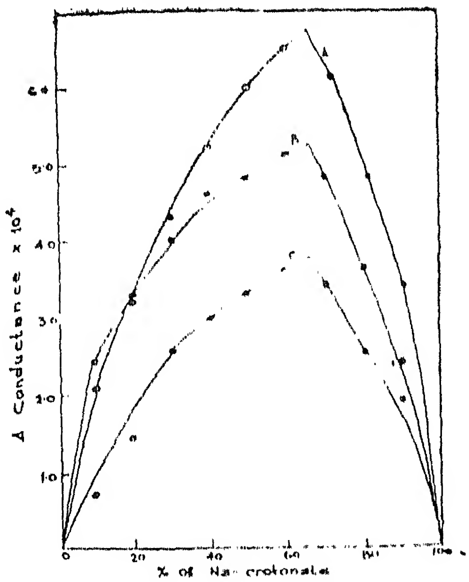


Fig. 2 Composition
Lm - Ligand
A M/60 - M/60
B M/80 - M/80
C M/100 M/100

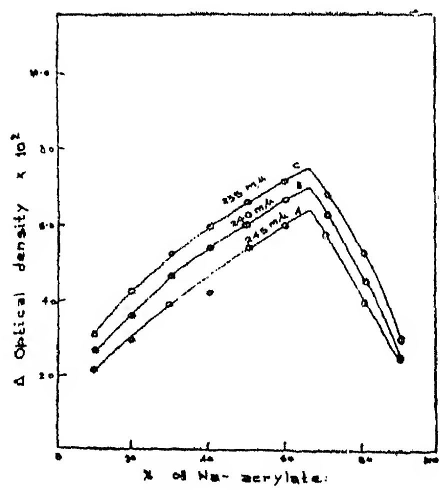


Fig. 3. Composition.

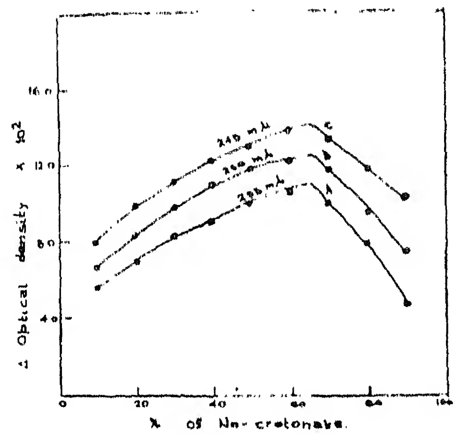


Fig. 4. Composition.

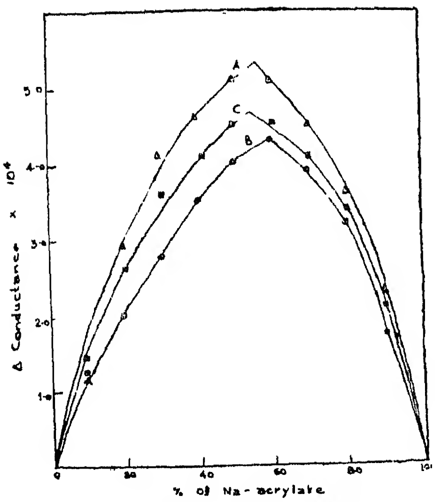


Fig 5 Dissociation Constant

	La -	Ligand
A.	M/100 -	M/40
B.	M/100 -	M/50
C.	M/120 -	M/40

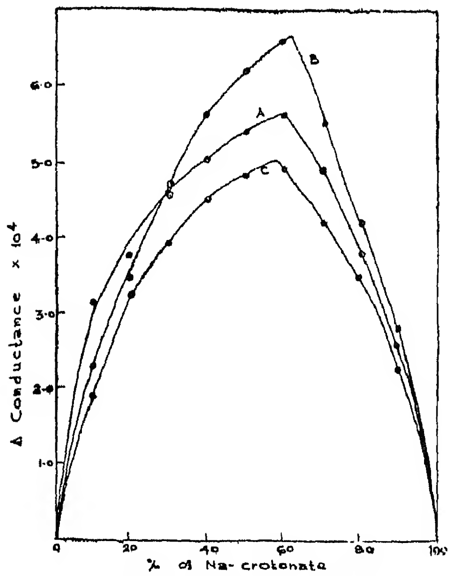


Fig. 6. Dissociation Constant.

	La	- Ligand
A	M/100 -	M/50
B.	M/80 -	M/50
C.	M/120 -	M/50

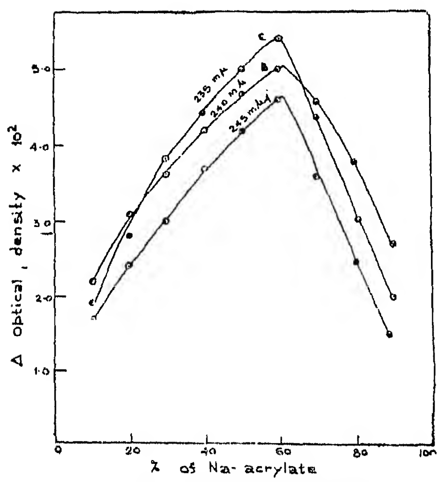


Fig 7. Dissociation Constant

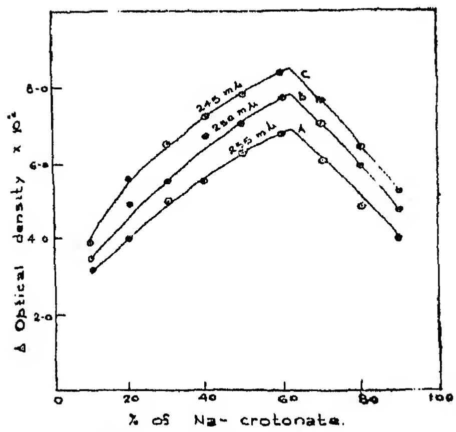


Fig. 8. Dissociation Constant.

For confirming the results obtained by the conductivity study the same method and procedure as described above was followed except that only one concentration and three specific wave lengths (Figs 3, 4) were employed for the optical density measurements. The blank solutions of the metal ion were not prepared since their absorbance was found to be nil Quartz cells of 0·5 mm. diameter were used.

DISSOCIATION CONSTANT

Other procedures remaining the same as in the determination of composition, three non-equimolar solutions of the metal and the ligand were used in the conductivity method, and one non-equimolar solution to be studied at three different wave lengths in the optical density method was employed.

The following equation of Job, which holds good only for 1 : 2 complexes, was used for the calculation of the dissociation constants of the complexes :

$$K = \frac{C^2\,[(p+2)\,x-2]^3}{(p-1)^2\,(2-3x)}$$

Where K=dissociation constant of the complex, C=the concentration of the metal ion, p=the ratio of the concentrations of the ligand and the metal ion, and x=fraction of the ligand used at the point of maxima. Figs. 5–8 incorporate the data.

DISCUSSION

Conductometric and spectrophotometric studies (Figs. 1–4) show the formation of only one complex of 1 : 2 metal to ligand ratio in each system. A study of the variation of optical density with wave length using mixtures of various compositions revealed that the same maxima were obtained in each case. This supports the above results.

Dissociation constants were found to be 4·26 × 10⁻⁵ and 5·52 × 10⁻⁵ for acrylate and crotonate complexes respectively, as indicated by Figs. 5–8. Crotonic acid is obtained by the introduction of one methyl group in acrylic acid which may cause some steric effects owing to which the stability of the complex may decrease as shown by their dissociation constants.

The following structures have been proposed for these complexes :

$$\left[\begin{array}{l} CH_2{=}CH-COO \\ CH_2{=}CH-COO \end{array}\right\rangle La\Big]^{+}$$

Lanthanum acrylate

$$\left[\begin{array}{l} CH_3-CH{=}CH-COO \\ CH_3-CH{=}CH-COO \end{array}\right\rangle La\Big]^{+}$$

Lanthanum crotonate.

REFERENCES

1. Andrew and Keeper, *J. Am. Chem. Soc.* **71** : 2381, (1949).
2. Kolthoff and Elmquist, *J. Am. Chem. Soc.* 53 : 1225, (1931).

THEOREMS CONCERNING THE NÖRLUND SUMMABILITY OF DERIVED FOURIER SERIES

By

LAL MANI TRIPATHI

Department of Mathematics, University of Allahabad, Allahabad (India)

[Received on 31st January, 1965]

ABSTRACT

Young (1914) and Riesz (1923) proved some theorems on the summability (C, α), $0 < \alpha \leqslant 1$, of the derived series of a Fourier series. The author (1963) studied the same series in the context of its harmonic summability. The object of this paper is to extend these theorems to certain types of Nörlund means. Since it is well known that the summability (C, α), $(\alpha > 0)$ and the harmonic summability are special cases of summability (N, p_n), many previously known results can be deduced from the theorems proved in this paper.

§ 1. Let $\{ s_n \}$ denote the sequence of partial sums of a given infinite series Σa_n, and $\{ p_n \}$ be a sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + \ldots + p_n \neq 0.$$

The sequence-to-sequence transformation

$$t_n = \sum_{\nu=1}^{n} p_{n-\nu} s_\nu / P_n, \tag{1·1}$$

defines the sequence $\{ t_n \}$ of Nörlund means[2], or simply the (N, p_n) means of the sequence $\{ s_n \}$, generated by the sequence of coefficients $\{ p_n \}$. The series Σa_n, or the sequence $\{ s_n \}$, is said to be summable (N, p_n) to the sum s, if

$$\lim_{n \to \infty} t_n = s.$$

The conditions for the regularity of the method of summability (N, p_n) defined by (1·1) are

$$\lim_{n \to \infty} p_n / P_n = 0, \tag{1·2}$$

and

$$\sum_{k=0}^{n} | p_k | = O (| P_n |), \text{ as } n \to \infty. \tag{1·3}$$

If p_n is real and non-negative, (1·3) is automatically satisfied and then (1·2) is the necessary and sufficient condition for the regularity of the method [1, § 4·2, Theorem 16].

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}, \ (\alpha > 0), \tag{1·4}$$

the Nörlund mean reduces to the familiar Cesàro mean of order α [1, § 5·13].

And for the value for which

$$p_n = \frac{1}{n+1}, \tag{1·5}$$

and, therefore,

$$P_n \sim \log n, \text{ as } n \to \infty,$$

t_n transforms into harmonic mean, defined by M. Riesz[5].

§ 2. We shall consider a function $f(t)$ of bounded variation, integrable in the sense of Lebesgue and periodic with period 2π.

If

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

then $f(t)$ generates the Fourier-Lebesgue series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \tag{2·1}$$

The series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} n\, B_n(t), \tag{2·2}$$

which is obtained by differentiating (2·1) term by term is called the first derived series or the derived Fouries series of $f(t)$.

We write

$$g(t) \equiv g(t, x) = f(x+t) - f(x-t) - 2t\, f'(x),$$

$$G(t) \equiv \int_0^t | dg(u) |,$$

$$K_n(t) = \sum_{\nu=0}^{n} p_{n-\nu} \sin (\nu + \tfrac{1}{2})t,$$

$$P_n = \sum_{\nu=0}^{n} p_\nu \sim P(n), \ p_n = p(n).$$

§ 3. The (C, α) summability of the derived series (2·2) corresponding to a function $f(x)$ of bounded variation has been considered by Young[7] and Riesz[4]. They have proved :

Theorem A. The derived series corresponding to a function $f(x)$ of bounded variation is summable (C, α), $0 < \alpha \leqslant 1$ to $f'(x)$ at the point x for which

$$G(t) \equiv \int_0^t | \, dg(u) \, | = o(t), \tag{3·1}$$

i.e. for almost every x.

Regarding the harmonic summability of the derived series (2·2) the present author[6] has, very recently, established the following theorem :

Theorem B. The derived series corresponding to a function $f(x)$ of bounded variation is summable by harmonic means to the sum $f'(x)$ at a point x at which

$$G(t) \equiv \int_0^t | \, dg(u) \, | = o\left(t/\log \frac{1}{t} \right), \text{ as } t \to 0 \tag{3·2}$$

The object of this paper is to study the derived series (2·2) by the Nörlund summability method which is more comprehensive than either of the two summability methods, namely, Cesàro and harmonic. Thus our result will extend the above Theorems A and B for the more general summability method of Nörlund.

§ 4. In what follows we establish the following theorems :

Theorem 1. *If $f(x)$ be a function of bounded variation and*

$$\int_{1/n}^{\delta} G(u) \left| \frac{d}{du} \frac{F(1/u)}{u} \right| du = o\,(P_n), \tag{4·1}$$

as $n \to \infty$, then the derived series $\sum_{n=1}^{\infty} n \, B_n(t)$, at $t = x$, is summable (N, p_n) to $f'(x)$, where $p(u)$ is a positive and monotonic decreasing function for $u \geqslant 0$ such that $p_n \equiv (p_n)$ and

$$F(u) \equiv \int_0^u p(s) \, ds \to \infty,$$

as $u \to \infty$.

With slight change, Theorem 1 may be restated in the following modified form :

Theorem 2. *If $f(x)$ be a function of bounded variation and*

$$\int_0^t \frac{P(1/u)}{u} \left| \, dg(u) \, \right| = o \, \{ P(1/t) \} \; ; \tag{4·3}$$

as $t \to 0$, then the derived series $\sum_{n=1}^{\infty} n \, B_n(t)$, at $t = x$, is summable (N, p_n) to $f'(x)$, where $p(u)$ is positive and monotonic decreasing function for $u \geqslant 0$ such that $p_n \equiv p(n)$ and

$$P(u) \equiv \int_0^u p(s)\, ds \to \infty,$$

as $u \to \infty$.

By an appeal to Lemma 1, we observe that if $p(u)$ is a positive decreasing function for $u \geqslant 0$ and the condition (4·2) together with the condition

$$\int_1^u \frac{P(s)}{s}\, ds = O\{P(u)\}, \text{ as } u \to \infty, \tag{4·4}$$

holds, then the conditions (4·1) and (3·1) become equivalent.

Also if, in (4·2), $p(u)$ is defined by (1·4) then the corresponding $P(u)$ satisfies (4·4). Hence the Theorem A easily follows from Theorem 1.

Again if we choose $p(u) = \dfrac{1}{u+1}$, then from (4·2) it follows that

$$P(u) \sim \log(u), \text{ as } u \to \infty.$$

Hence for a sufficiently small $\delta > 0$ and $n \to \infty$,

$$\begin{aligned}\int_{1/n}^{\delta} G(u) \left| \frac{d}{du} \frac{P(1/u)}{u} \right| du &= O(1) \int_{1/n}^{\delta} G(u) \frac{\log(1/u)}{u^2}\, du \\ &= o(1) \int_{1/n}^{\delta} \frac{u}{\log(1/u)} \frac{\log(1/u)}{u^2}\, du \\ &= o(\log n) \\ &= o(P_n),\end{aligned}$$

by an application of (3·2). Therefore (3·2) and (1·5) together imply (4·1) and consequently Theorem B is a particular case of Theorem 1.

§ 5. In order to prove the theorems, we use the following lemmas.

Lemma 1.[3] *If $p(u)$ is a positive and monotonic decreasing function for $u > 0$ such that $\int_0^u p(s)\, ds \equiv P(u) \to \infty$, as $u \to \infty$, then the condition*

$$\int_{1/n}^{\delta} G(u) \left| \frac{d}{du} \frac{P(u)}{u} \right| du = o(P_n), \text{ as } n \to \infty,$$

implies

$$G(t) \equiv \int_0^t |\, dg(u)\, | = o(t), \text{ as } t \to 0.$$

Moreover, if $P(u)$ satisfies the additional condition (4·4), then the above conditions are equivalent.

Lemma 2. [3, § 2, (12)]. *If $p(u)$ satisfies the same conditions as in Lemma 1, then for $1/n \leqslant t \leqslant \delta$.*

$$\left| \frac{K_n(t)}{\sin\frac{1}{2}t} \right| < A \frac{P(1/t)}{t},$$

where A is a positive constant.

Lemma 3.—*Under the same conditions as in Lemma 1, the condition*

$$\int_0^t \frac{P(1/u)}{u} \mid dg(u) \mid = o(P(1/t)), \text{ as } t \to 0,$$

implies

$$\int_0^t \mid dg(u) \mid = o(t), \text{ as } t \to 0.$$

As $P(1/u)/u$ is monotonic decreasing for $u > 0$,

$$\frac{P(1/t)}{t} \int_0^t \mid dg(u) \mid < \int_0^t \frac{P(1/u)}{u} \mid dg(u) \mid,$$

and hence

$$\frac{P(1/t)}{t} \int_0^t \mid dg(u) \mid = o\{P(1/t)\},$$

i.e. $$\int_0^t \mid dg(u) \mid = o(t).$$

§ 6. **Proof of the Theorem 1.** By Lemma 1, (4·1) implies

$G(t) \equiv o(t)$, as $t \to 0$,

we shall make frequent use of this fact in the proof of this theorem.

Denoting by $s_n'(x)$ the sum of the first n terms of the series (2·2), at $t = x$, we get

$$\begin{aligned}
s_n'(x) &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{d}{dx} \frac{\sin(n+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)} \right\} f(u)\, du \\
&= -\frac{1}{2\pi} \int_0^{2\pi} f(u) \left\{ \frac{d}{du} \frac{\sin(n+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)} \right\} du \\
&= -\frac{1}{2\pi} \int_0^{\pi} \{ f(x+t) - f(x-t) \} \left\{ \frac{d}{dt} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \right\} dt \\
&= \frac{1}{2\pi} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, d\{ f(x+t) - f(x-t) \} \\
&= \frac{1}{2\pi} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dg(t) + f'(x).
\end{aligned}$$

Hence

$$s_n'(x) - f'(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dg(t)$$

$$= \frac{1}{2\pi}\int_0^{\delta} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dg(t) + o(1).$$

Taking the Nörlund mean of both the sides, and denoting by $T_n(x)$ the Nörlund mean of $s_n'(x) - f'(x)$ and remembering that the method of summability is regular, we have

$$T_n(x) = \frac{1}{2\pi P_n}\int_0^{\delta} \frac{K_n(t)}{\sin\frac{1}{2}t}\, dg(t) + o(1)$$

$$= \frac{1}{2\pi P_n}\left\{\int_0^{1/n} + \int_{1/n}^{\delta}\right\}\frac{K_n(t)}{\sin\frac{1}{2}t}\, dg(t) + o(1),$$

$$= \alpha_n + \beta_n + o(1),$$

say.

Now

$$|\alpha_n| \leqslant \frac{1}{2\pi P_n}\int_0^{1/n} \left|\frac{\sum_{\nu=0}^{n} P_{n-\nu}\sin(\nu+\frac{1}{2})t}{|\sin\frac{1}{2}t|}\right| |dg(t)|$$

$$\leqslant \frac{(2n+1)}{2\pi}\int_0^{1/n} |dg(t)|$$

$$= (2n+1)\cdot o\left(\frac{1}{n}\right)$$

$$= o(1).$$

Again, for $t > 0$, we have by Lemma 2

$$\left|\frac{K_n(t)}{\sin\frac{1}{2}t}\right| < A\,\frac{P(1/t)}{t}.$$

Therefore

$$|\beta_n| \leqslant \frac{1}{2\pi P_n}\int_{1/n}^{\delta}\left|\frac{K_n(t)}{\sin\frac{1}{2}t}\right| |dg(t)|$$

$$< \frac{A}{2\pi P_n}\int_{1/n}^{\delta} \frac{P(1/t)}{t}\, |dg(t)|$$

$$\leqslant \frac{A}{2\pi P_n}\left[G(t)\,P(1/t)\,\frac{1}{t}\right]_{1/n}^{\delta} + \frac{A}{2\pi P_n}\int_{1/n}^{\delta} G(t)\left|\frac{d}{dt}\,\frac{P(1/t)}{t}\right| dt$$

$$\leqslant \frac{A}{2\pi P_n}\left[\frac{G(\delta)}{\delta}\,P(1/\delta) + G(1/n)\; nP(n)\right] + \frac{A}{2\pi P_n}\,[o(P_n)]$$

$$= o(1),$$

since $P(1/\delta) < P(n) \sim P_n$.

This completes the proof of Theorem 1.

Proof of Theorem 2. Proceeding as in the proof of Theorem 1, we obtain

$$T_n(x) = \alpha_n + \beta_n + o(1).$$

We observe from Theorem 1 that

$$\alpha_n = o(1),$$

as condition (4·3) implies (3·1), by Lemma 3.

Now

$$\begin{aligned} |\beta_n| &\leqslant \frac{1}{2\pi P_n} \int_{1/n}^{\delta} \frac{|K_n(t)|}{|\sin\frac{1}{2}t|} \, |dg(t)| \\ &< \frac{A}{2\pi P_n} \int_{1/n}^{\delta} \frac{P(1/t)}{t} \, |dg(t)| \\ &\leqslant \frac{A}{2\pi P_n} \int_{0}^{\delta} \frac{P(1/t)}{t} \, |dg(t)| + \frac{A}{2\pi P_n} \int_{0}^{1/n} \frac{P(1/t)}{t} \, |dg(t)| \\ &\leqslant \frac{B}{P_n} \{P(1/\delta)\} + o\,(1), \end{aligned}$$

by virtue of Lemma 2 and hypothesis of the Theorem 2. A and B are positive constants.

Hence

$$\beta_n = o(1).$$

This completes the proof of Theorem 2.

I am indebted to Professors R. S. Mishra and P. L. Srivastava for their generous encouragement.

REFERENCES

1. Hardy, G. H. Divergent series, *Oxford*, (1949).
2. Nörlund, N. E. Sur une application des fonctions permutables, *Lunds Universitets Arsskrift*, (2), **16** : 1–10, (1919).
3. Rajagopal, C. T. On the Nörlund summability of Fourier series, *Proc. Camb. Phil. Soc.*, **59** : 47–53, (1963).
4. Riesz, M. Sur la sommation des series de Fourier, *Acta Szeged*, **1** : 104–113, (1923).
5. Riesz, M. Sur l'equivalence de certaines methods de sommation, *Proc. London Math. Soc.*, (2), **22** : 412-419, (1924).
6. Tripathi, L. M. On the harmonic summability of the derived Fourier series and its conjugate series, *Proc. Nat. Acad. Sci. India*, **33** : 443–454, (1963).
7. Young, W. H. The usual convergence of a class of trigonometrical series, *Proc. Lond, Math. Soc.*, **13** : 13–28, (1914).

SOLVABLE CASES OF THE GENERAL SIXTH AND EIGHTH DEGREE EQUATIONS

By

D. RAMESWAR RAO

Department of Mathematics, Osmania University, Hyderabad

[Received on 23rd February, 1965]

ABSTRACT

It is a well-known fact that a general equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0, \text{ for } n \geq 5,$$

has no general method of attack Besides Abel's demonstration regarding the impossibility of solving any such equations by a general method, is generally accepted by mathematicians Although, there are special cases in which such equations can be solved by particular methods, those are not appearing to have been dealt. The object of this note is to discuss the solutions of

(*i*) $x^6 - px^4 + qx^3 - rx^2 + s = 0$ and $x^8 - px^6 + qx^4 - rx^2 + s = 0$

when $p(q^2 - 4s) = r^2$

(*ii*) $x^6 + Qx^3 - Px^2 - Rx + S = 0$ and $x^8 + Qx^4 - Px^2 - Rx + S = 0$

when $P(Q^2 - 4S) = R^2$

Problem. Find the roots of the following equations

(i) $x^6 - px^4 + qx^3 - rx^2 + s = 0$, (*ii*) $x^6 + Qx^3 - Px^2 - Rx + S = 0$

(*iii*) $x^8 - px^6 + qx^4 - rx^2 + s = 0$, (*iv*) $x^8 + Qx^4 - Px^2 - Rx + S = 0$

when $p(q^2 - 4s) = r^2$ and $P(Q^2 - 4S) = R^2$

Solution :

$$x^6 - px^4 + qx^3 - rx^2 + s = 0$$

$$\therefore \quad x^6 + qx^3 + s = px^4 + rx^2$$

$$\therefore \ 4px^6 + 4pqx^3 + 4ps + r^2 = (2px^2 + r)^2 \qquad \ldots . (i)$$

$$p(q^2 - 4s) = r^2 \qquad \ldots . (ii)$$

$$\therefore \ p^{\frac{1}{2}} q = (4ps + r^2)^{\frac{1}{2}} \qquad \ldots . (iii)$$

$$\therefore \ 4pq = 2 \cdot 2p^{\frac{1}{2}} (4ps + r^2)^{\frac{1}{2}} \qquad \ldots . (iv)$$

$\therefore$ From (*i*) and (*iv*)

$$\{2p^{\frac{1}{2}} x^3 + (4ps + r^2)^{\frac{1}{2}} \}^2 = (2px^2 + r)^2 \qquad \ldots . (v)$$

$\therefore$ From (*ii*), (*iii*) and (*v*)

$$(2x^3 + q)^2 = \{2p^{\frac{1}{2}} x^2 + (q^2 - 4s)^{\frac{1}{2}} \}^2$$

Hence the six roots are given by

$$2x^3 + 2p^{\frac{1}{2}} x^2 + \{q + (q^2 - 4s)^{\frac{1}{2}}\} = 0 \qquad \ldots.(vi)$$

and $$2x^3 - 2p^{\frac{1}{2}} x^2 + \{q - (q^2 - 4s)^{\frac{1}{2}}\} = 0 \qquad \ldots.(vii)$$

Put $x = \mathrm{X} - \frac{p^{\frac{1}{2}}}{3}$ in (vi).

$$x^3 - \frac{px}{3} + \left[\frac{27\ \{q + (q^2 - 4s)^{\frac{1}{2}}\} - 4p^{\frac{3}{2}}}{54}\right] = 0$$

Hence the roots of (vi) are given by

$$x = \mathrm{X} - \frac{p^{\frac{1}{2}}}{3} = u + v - \frac{p^{\frac{1}{2}}}{3} \text{ or } uw + vw^2 - \frac{p^{\frac{1}{2}}}{3}$$

$$\text{or } uw^2 + vw - \frac{p^{\frac{1}{2}}}{3}$$

where $u = \left\{-\frac{R}{2} + \sqrt{\frac{R^2}{4} - \frac{p^3}{27^2}}\right\}^{\frac{1}{3}},\ v = \left\{\frac{-R}{2} - \sqrt{\frac{R^2}{4} - \frac{p^3}{27^2}}\right\}^{\frac{1}{3}}$

where $R = \left[\frac{27\ \{q + (q^2 - 4s)^{\frac{1}{2}}\} - 4p^{\frac{3}{2}}}{54}\right]$

Similarly the roots of (vii) are given by

$$x = x' + \frac{p^{\frac{1}{2}}}{3} = u' + v' + \frac{p^{\frac{1}{2}}}{3} \text{ or } u'w + v'w^2 + \frac{p^{\frac{1}{2}}}{3}$$

$$\text{or } u'w^2 + v'w + \frac{p^{\frac{1}{2}}}{3}$$

where $u' = \left\{\frac{-R'}{2} + \sqrt{\frac{R'^2}{4} - \frac{p^3}{27^2}}\right\}^{\frac{1}{3}},\ v' = \left\{\frac{-R'}{2} - \sqrt{\frac{R'^2}{4} - \frac{p^3}{27^2}}\right\}^{\frac{1}{3}}$

where $R' = \frac{27\ \{q - (q^2 - 4s)^{\frac{1}{2}}\} - 4p^{\frac{3}{2}}}{54}$

Aliter :

$$x^6 + qx^3 + s = px^4 + rx^2$$

$$\therefore\ 4x^6 + 4qx^3 + 4s = 4px^4 + 4rx^2$$

$$\therefore\ (2x^3 + q)^2 = 4px^4 + 4rx^2 + q^2 - 4s \qquad \ldots.(1)$$

But $p\ (q^2 - 4s) = r^2$

$$\therefore\ 2{\cdot}2p^{\frac{1}{2}}\ (q^2 - 4s)^{\frac{1}{2}} = 4r \qquad \ldots.(2)$$

$\therefore$ From (1) and (2) $\qquad (2x^3 + q)^2 = \{2p^{\frac{1}{2}} x^2 + (q^2 - 4s)^{\frac{1}{2}}\}^2$

Hence the solution of the problem follows as above.

Similarly, the roots of the equation

$$x^6 + Qx^3 - Px^2 - Rx + S = 0,$$

can be obtained by solving the following two cubics

$$2x^3 + 2P^{\frac{1}{2}} x + \{ Q + (Q^2 - 4S)^{\frac{1}{2}} \} = 0$$

$$2x^3 - 2P^{\frac{1}{2}} x + \{ Q - (Q^2 - 4S)^{\frac{1}{2}} \} = 0$$

By a similar manipulation, the roots of the equations

$$x^8 - px^6 + qx^4 - rx^3 + s = 0$$

$$x^8 + Qx^4 - Px^2 - Rx + S = 0$$

can be obtained by solving the following two pairs of biquadratics respectively

$$2x^4 + 2p^{\frac{1}{2}} x^3 + \{q + (q^2 - 4s)\}^{\frac{1}{2}} = 0$$

$$2x^4 - 2p^{\frac{1}{2}} x^3 + \{q - (q^2 - 4s)^{\frac{1}{2}} \} = 0$$

and

$$2x^4 + 2P^{\frac{1}{2}} x + \{ Q + (Q^2 - 4s)^{\frac{1}{2}} \} = 0$$

$$2x^4 - 2P^{\frac{1}{2}} x + \{Q - ((Q^2 - 4S)^{\frac{1}{2}} \} = 0$$

CYLINDER FUNCTIONS OF SEVERAL ARGUMENTS

By

S. L. GUPTA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 18th March, 1965]

ABSTRACT

Here the two types of cylinder functions of n arguments are defined and their certain results for $n = 2$ have been given.

1. The functions $J^{(1)}_{\mu}\{x_n\}$ and $J^{(2)}_{\mu}\{x_n\}$ are generalisations of the Bessel function $J_{\mu}(x_1)$ to functions of n arguments $x_1, \ldots, x_n$ and they are defined by the equalities :

$$J^{(1)}_{\mu}\{x_n\} = \sum^{\infty} J_{t_1}(x_1) \ldots J_{t_n}(x_n), \tag{1}$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu$

$$J^{(2)}_{\mu}\{x_n\} = \sum^{\infty} J_{t_1}(x_1)\, I_{t_2}(x_2)\, J_{t_3}(x_3) \ldots \tag{2}$$

$t_1, \ldots, t_n = -\infty$ except t_r, which is given by $t_1 + \ldots + nt_n = \mu$

the last term within the summation in (2) is $J_{t_n}(x_n)$ or $I_{t_n}(x_n)$ according as n is an odd or even integer.

The subscript r may have any integral value from 1 to n.

The other kinds of the generalised Bessel functions $Y^{(1,2)}_{\mu}\{x_n\}$, $H^{(1,2),(1)}_{\mu}\{x_n\}$ and $H^{(1,2),(2)}_{\mu}\{x_n\}$ are the generalisations of the functions $Y_{\mu}(x_1)$, $H^{(1)}_{\mu}(x_1)$ and $H^{(2)}_{\mu}(x_1)$. The superscript (1, 2) stands either for (1) to denote the first type of the generalised function or for (2) to denote the second type of the generalised function.

These generalised functions are defined by the equalities :

$$Y^{(1)}_{\mu}\{x_n\} = \frac{J^{(1)}_{\mu}\{x_n\}\cos\mu\pi - J^{(1)}_{-\mu}\{(-1)^{n+1}x_n\}}{\sin\mu\pi},$$

$$Y^{(2)}_{\mu}\{x_n\} = \frac{J^{(2)}_{\mu}\{x_n\}\cos\mu\pi - J^{(2)}_{-\mu}\{x_n\}}{\sin\mu\pi}; \tag{3}$$

$$H^{(1),(1)}_{\mu}\{x_n\} = J^{(1)}_{\mu}\{x_n\} + i\,Y^{(1)}_{\mu}\{x_n\},\quad H^{(2),(1)}_{\mu}\{x_n\} = J^{(1)}_{\mu}\{x_n\} - i\,Y^{(1)}_{\mu}\{x_n\}; \tag{4}$$

$$H^{(1),(2)}_{\mu}\{x_n\} = J^{(2)}_{\mu}\{x_n\} + i\,Y^{(2)}_{\mu}\{x_n\},\quad H^{(2),(2)}_{\mu}\{x_n\} = J^{(2)}_{\mu}\{x_n\} - i\,Y^{(2)}_{\mu}\{x_n\}; \tag{5}$$

the numerators and the denominators in (3) both vanish when $\mu = 0, \pm 1$ $\pm 2, \ldots .$

In these cases the functions are defined as the limits of the ratio.

In a recent paper[2] it has been shown that each of the function $J_{\mu}^{(1,2)}\{x_n\}$ satisfies $n+1$ recurrence formulae given by

$$\mu J_{\mu}^{(1)}\{x_n\} = \tfrac{1}{2}x_1 (J_{\mu-1}^{(1)}\{x_n\} + J_{\mu+1}^{(1)}\{x_n\}) + \ldots + \tfrac{1}{2}n\, x_n (J_{\mu-n}^{(1)}\{x_n\} + J_{\mu+n}^{(1)}\{x_n\}), \tag{6}$$

$$\frac{\partial J_{\mu}^{(1)}\{x_n\}}{\partial x_p} = \tfrac{1}{2}(J_{\mu-p}^{(1)}\{x_n\} - J_{\mu+p}^{(1)}\{x_n\}); \tag{7}$$

and

$$\mu J_{\mu}^{(2)}\{x_n\} = \tfrac{1}{2}x_1 (J_{\mu-1}^{(2)}\{x_n\} + J_{\mu+1}^{(2)}\{x_n\}) + \ldots + \tfrac{1}{2}n\, x_n (J_{\mu-n}^{(2)}\{x_n\} + (-1)^{n+1} J_{\mu+n}^{(2)}\{x_n\}) \tag{8}$$

$$\frac{\partial J_{\mu}^{(2)}\{x_n\}}{\partial x_p} = \tfrac{1}{2}(J_{\mu-p}^{(2)}\{x_n\} + (-1)^p J_{\mu+p}^{(2)}\{x_n\}); \tag{9}$$

respectively, where $p = 1, \ldots, n$.

To obtain the recurrence formulae satisfied by the other kinds of the functions, let us first suppose that $\mu \neq 0, \pm 1, \pm 2, \ldots$ then from (6), we have

$$\begin{aligned}
&\frac{\mu J_{\mu}^{(1)}\{x_n\}\cos\mu\pi - \mu J_{-\mu}^{(1)}\{(-1)^{n+1}x_n\}}{\sin\mu\pi} = [\tfrac{1}{2}x_1 (J_{\mu-1}^{(1)}\{x_n\} + J_{\mu+1}^{(1)}\{x_n\}) \\
&\qquad + \ldots \tfrac{1}{2}n\, x_n (J_{\mu-n}^{(1)}\{x_n\} + J_{\mu+n}^{(1)}\{x_n\})]\cot\mu\pi \\
&\qquad + [\tfrac{1}{2}x_1 (J_{-\mu-1}^{(1)}\{(-1)^{n+1}x_n\} + J_{-\mu+1}^{(1)}\{(-1)^{n+1}x_n\}) \ldots \\
&\qquad + \tfrac{1}{2}(-1)^{n+1} n x_n (J_{-\mu-n}^{(1)}(-1)^{n+1}x_n\} + J_{-\mu+n}^{(1)} \\
&\qquad \{(-1)^{n+1}x_n\})]\operatorname{cosec}\mu\pi \\
&= \tfrac{1}{2}x_1 [J_{\mu-1}^{(1)}\{x_n\}\cot(\mu-1)\pi - J_{-\mu+1}^{(1)}\{(-1)^{n+1}x_n\}\operatorname{cosec}(\mu-1)\pi) \\
&\qquad + J_{\mu+1}^{(1)}\{x_n\}\cot(\mu+1)\pi - J_{-\mu-1}^{(1)}\{(-1)^{n+1}x_n\}\operatorname{cosec}(\mu+1)\pi] \\
&+ \ldots + \tfrac{1}{2}n\, x_n [J_{\mu-n}^{(1)}\{x_n\}\cot(\mu-n)\pi - J_{-\mu+n}^{(1)}\{(-1)^{n+1}x_n\}\operatorname{cosec}(\mu-n)\pi \\
&\qquad + J_{\mu+n}^{(1)}\{x_n\}\cot(\mu+n)\pi - J_{-\mu-n}^{(1)}\{x_n\}\operatorname{cosec}(\mu+n)\pi].
\end{aligned}$$

Therefore, from (3) we find that

$$\mu\, \gamma^{(1)}_{\mu}\, \{x_n\} = \tfrac{1}{2} x_1 \left(\gamma^{(1)}_{\mu - 1}\, \{x_n\} + {}^{(1)}_{\mu + 1}\, \{x_n\} + \ldots + \tfrac{1}{2} n\, x_n \right)$$
$$\left({}^{(1)}_{\mu - n}\, \{x_n\} + \gamma^{(1)}_{\mu + n}\, \{x_n\} \right), \tag{10}$$

And it is easy to see that

$$\frac{\partial\, \gamma^{(1)}_{\mu}\, \{x_n\}}{\partial x_p} = \tfrac{1}{2} \left(\gamma^{(1)}_{\mu - p}\, \{x_n\} - \gamma^{(1)}_{\mu + p}\, \{x_n\} \right). \tag{11}$$

We shall now prove that the formulae (10) and (11) are also true when $\mu = 0, \pm 1, \pm 2, \ldots$; since $J_{\mu}(x)$ and its derivatives are continuous functions of μ, therefore, $J^{(1)}_{\mu}\, \{x_n\}$ and its derivatives are also continuous functions of μ, so that $\gamma^{(1)}_{\mu}\, \{x_n\}$ and its derivatives are also continuous functions of μ and hence (10) and (11) are also true for $\mu = 0, \pm 1, \pm 2, \ldots$.

Thus it is found that $\gamma^{(1)}_{\mu}\, \{x_n\}$ satisfies the same recurrence formulae as $J^{(1)}_{\mu}\, \{x_n\}$; and since $H^{(1),(1)}_{\mu}\{x_n\}, H^{(2),(1)}_{\mu}\{x_n\}$ are linear functions with constant coefficients of these functions, it follows that $H^{(1,2),(1)}_{\mu}\, \{x_n\}$ will satisfy the same recurrence formulae (6) and (7).

Similarly, we find that $\gamma^{(2)}_{\mu}\, \{x_n\}$, $H^{(1,2),(2)}_{\mu}\, \{x_n\}$ satisfy the same recurrence formulae (8) and (9) which are satisfied by the function $J^{(2)}_{\mu}\, \{x_n\}$.

As we have seen that the three kinds of the generalised Bessel functions of the first type satisfy a set of recurrence formulae of the form (6) and (7), on analogy from the definition of the cylinder function $C_{\mu}(x)$ we shall, therefore, define a cylinder function of the first type and of unrestricted arguments $x_1, \ldots, x_n$ and order μ by $C^{(1)}_{\mu}\, \{x_n\}$ which satisfies the following recurrence formulae

$$\mu\; C^{(1)}_{\mu}\, \{x_n\} = \tfrac{1}{2} x_1 \left(C^{(1)}_{\mu - 1}\, \{x_n\} + C^{(1)}_{\mu + 1}\, \{x_n\}\right) + \ldots +$$
$$+ \tfrac{1}{2} n\, x_n \left(C^{(1)}_{\mu - n}\, \{x_n\} + C^{(1)}_{\mu + n}\, \{x_n\} \right), \tag{12}$$

$$\frac{\partial\, C^{(1)}_{\mu}\, \{x_n\}}{\partial x_p} = \tfrac{1}{2} \left(C^{(1)}_{\mu - p}\, \{x_n\} - C^{(1)}_{\mu + p}\, \{x_n\} \right). \tag{13}$$

Similarly, we can define a second type of cylinder function $C^{(2)}_{\mu}\, \{x_n\}$ of the unrestricted arguments $x_1, \ldots, x_n$ and the order μ which satisfies the recurrence formulae

$$\mu\, C^{(2)}_{\mu}\{x_n\} = \tfrac{1}{2}x_1\left(C^{(2)}_{\mu-1}\{x_n\} + C^{(2)}_{\mu+1}\{x_n\}\right) + \ldots +$$
$$+ \tfrac{1}{2}n x_n\left(C^{(2)}_{\mu-n}\{x_n\} + (-1)^{n+1}\, C^{(2)}_{\mu+n}\{x_n\}\right), \quad (14)$$

$$\frac{\partial\, C^{(2)}_{\mu}\{x_n\}}{\partial x_p} = \tfrac{1}{2}\left(C^{(2)}_{\mu-p}\{x_n\} + (-1)^p\, C^{(2)}_{\mu+p}\{x_n\}\right). \quad (15)$$

These formulae are satisfied by each of the three kinds of Bessel functions of the second type.

It is interesting to note that $E^{(1)}_{2m}\{x_n\}$ and $E^{(2)}_{2m}\{x_n\}$ which are defined by [1, p. 154 ; 2]

$$E^{(1)}_{m}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} \sin\,(m\,\theta - x_1 \sin\theta - \ldots - x_n \sin n\,\theta)\, d\,\theta,$$

$$E^{(2)}_{m}\{x_n\} = \frac{1}{\pi}\int_0^{\pi} e^{x_2 \cos 2\,\theta + x_4 \cos 4\,\theta + \ldots}$$
$$\sin\,(m\,\theta - x_1 \sin\theta - x_3 \sin 3\,\theta - \ldots)\, d\,\theta,$$

are also examples of the first and the second type of cylinder functions of the unrestricted arguments $x_1, \ldots x_n$ but of restricted order $2m$, where m is an integer.

2. From the above definitions of the cylinder functions various results can be obtained, for simplicity let us consider the case when $n = 2$. So that from (12) and (14) we can write

$$x_2\, C^{(1)}_{\mu-2}(x_1, x_2) + \tfrac{1}{2}x_1\, C^{(1)}_{\mu-1}(x_1, x_2) + x_2\, C^{(1)}_{\mu}(x_1, x_2) = (2x_2 + \mu)\, C^{(1)}_{\mu}(x_1, x_2) -$$
$$- \left(x_2\, C^{(1)}_{\mu}(x_1, x_2) + \tfrac{1}{2}x_1\, C^{(1)}_{\mu+1}(x_1, x_2) + x_2\, C^{(1)}_{\mu+2}(x_1, x_2)\right)$$

$$x_2\, C^{(2)}_{\mu-2}(x_1, x_2) + \tfrac{1}{2}x_1\, C^{(2)}_{\mu-1}(x_1, x_2) - x_2\, C^{(2)}_{\mu}(x_1, x_2) = \mu\, C^{(2)}_{\mu}(x_1, x_2) -$$
$$- \left(x_2\, C^{(2)}_{\mu}(x_1, x_2) + \tfrac{1}{2}x_1\, C^{(2)}_{\mu+1}(x_1, x_2) - x_2\, C^{(2)}_{\mu+2}(x_1, x_2)\right),$$

and, therefore, by repeated application of these we, get

$$x_2\, C^{(1)}_{\mu-2}(x_1, x_2) + \tfrac{1}{2}x_1\, C^{(1)}_{\mu-1}(x_1, x_2) + x_2\, C^{(1)}_{\mu}(x_1, x_2) = (2x_2 + \mu)\, C^{(1)}_{\mu}(x_1, x_2) -$$
$$- (2x_2 + \mu + 2)\, C^{(1)}_{\mu+2}(x_1, x_2) + (2x_2 + \mu + 4)\, C^{(1)}_{\mu+4}(x_1, x_2) - \ldots, \quad (16)$$

$$x_2\, C^{(2)}_{\mu-2}(x_1, x_2) + \tfrac{1}{2}x_1\, C^{(2)}_{\mu-1}(x_1, x_2) - x_2\, C^{(2)}_{\mu}(x_1, x_2) = \mu\, C^{(2)}_{\mu}(x_1, x_2) -$$
$$- (\mu + 2)\, C^{(2)}_{\mu+2}(x_1, x_2) + (\mu + 4)\, C^{(2)}_{\mu+4}(x_1, x_2) - \ldots \quad (17)$$

It is simple to find that

$$\frac{2^k \partial^k C_\mu^{(1)}(x_1, x_2)}{\partial x_1{}^k} = C^{(1)}_{\mu-k}(x_1, x_2) - k\, C^{(1)}_{\mu-k+2}(x_1, x_2) + \frac{k(k-1)}{2!}$$
$$C^{(1)}_{\mu-k+4}(x_1\, x_2) - \ldots + (-1)^k\, C^{(1)}_{\mu+k}(x_1 \cdot x_2), \tag{18}$$

$$\frac{2^k \partial^k C_\mu^{(1)}(x_1\;\; x_2)}{\partial x_2{}^k} = C^{(1)}_{\mu-2k}(x_1, x_2) - k\, C^{(1)}_{\mu-2(k-2)}(x_1\;\; x_2) + \frac{k(k-1)}{2!}$$
$$C^{(1)}_{\mu-2(k-1)}(x_1, x_2) - \ldots + (-1)^k\, C^{(1)}_{\mu+2k}(x_1 \cdot x_2), \tag{19}$$

$$\frac{2^k \partial^k C_\mu^{(1)}(x_1 \cdot x_2)}{\partial x_2{}^k} = C^{(2)}_{\mu-2k}(x_1, x_2) + k\, C^{(2)}_{\mu-2(k-2)} \cdot (x_1 \cdot x_2) + \frac{k(k-1)}{2!}$$
$$C^{(2)}_{\mu-2(k-4)}(x_1, x_2) + \ldots + C^{(2)}_{\mu+2k}(x_1, x_2). \tag{20}$$

In (18) the first type of the cylinder functions may be replaced by the corresponding cylinder functions of the second type.

In particular when $x_1 = x_2 = x$, we have

$$\frac{2\mu}{x} C_\mu^{(1)}(x, x) = C^{(1)}_{\mu-1}(x, x) + C^{(1)}_{\mu+1}(x, x) + 2\,(C^{(1)}_{\mu-2}(x, x) + C^{(1)}_{\mu+2}(x, x)),$$

$$2\frac{d}{dx} C_\mu^{(1)}(x, x) = C^{(1)}_{\mu-2}(x, x) + C^{(1)}_{\mu-1}(x, x) - C^{(1)}_{\mu+1}(x, x) - C^{(1)}_{\mu+2}(x, x),$$

and

$$\frac{2\mu}{x} C_\mu^{(2)}(x, x) = C^{(2)}_{\mu-1}(x, x) + C^{(2)}_{\mu+1}(x, x) + 2\,(C^{(2)}_{\mu-2}(x, x) - C^{(2)}_{\mu+2}(x, x)),$$

$$\frac{2d}{dx} C_\mu^{(2)}(x, x) = C^{(2)}_{\mu-2}(x, x) + C^{(2)}_{\mu-1}(x, x) - C^{(2)}_{\mu+1}(x, x) + C^{(2)}_{\mu+2}(x, x).$$

After some reductions it is found that

$$4\frac{d^2}{dx^2} C_\mu^{(1)}(x, x) = -\frac{4}{x} C^{(1)}_{\mu-2}(x, x) + \left(\frac{2\mu-5}{2x} - \frac{33}{8}\right) C^{(1)}_{\mu-1}(x, x) +$$
$$\left(\frac{\mu(\mu+2)}{x^2} + \frac{\mu}{4x} - \frac{15}{2}\right) C_\mu^{(1)}(x, x) + \left(\frac{2\mu+1}{2x} - \frac{33}{8}\right) C^{(1)}_{\mu+1}(x, x), \tag{21}$$

$$4\frac{d}{dx}\left(C^{(1)}_{\mu-2}(x, x) + C^{(1)}_{\mu-1}(x, x) + C^{(1)}_{\mu+1}(x, x) + C^{(1)}_{\mu+2}(x, x)\right)$$
$$= \left(\frac{2(\mu-2)}{x} + \frac{1}{2}\right) C^{(1)}_{\mu-2}(x, x) + \left(\frac{3(\mu-1)}{x} + 2\right) C^{(1)}_{\mu-1}(x, x) -$$
$$\left(\frac{3(\mu+1)}{x} + 2\right) C^{(1)}_{\mu+1}(x, x) - \left(\frac{2(\mu+2)}{x} + \frac{1}{2}\right) C^{(1)}_{\mu+2}(x, x). \tag{22}$$

$$\frac{d}{dx}\left(\left(4+\frac{2x}{\mu}\right) C^{(1)}_{\mu-2}(x,x) - \left(2-\frac{x}{\mu}\right) C^{(1)}_{\mu-1}(x,x) - \left(2-\frac{x}{\mu}\right) C^{(1)}_{\mu+1}(x,x) + \left(4+\frac{2x}{\mu}\right) C^{(1)}_{\mu+2}(x,x) \right)$$

$$= \frac{2(\mu-2)}{x} C^{(1)}_{\mu-2}(x,x) - 3\, C^{(1)}_{\mu-1}(x,x) + 3\, C^{(1)}_{\mu+1}(x,x) - \frac{2(\mu+2)}{x} C^{(1)}_{\mu+2}(x,x). \tag{23}$$

Similar results for the function $C^{(2)}_{\mu}(x, x)$ can also be obtained.

As it has been found, the above definitions of the two types of the generalised cylinder functions provides a further scope [**1**, pp. 207–223] for studying various properties of the generalised Bessel functions.

REFERENCES

1. Gupta, S. L. *Ph. D. Thesis, University of Jodhpur*, (1964).
2. Gupta, S. L *Proc. Nat. Acad. Sci, India*, to appear.
3. Watson, G. N. *Theory of Bessel Functions*, p. 82, 2nd Ed., (1958).

QUASI-UNIFORM RADIAL OSCILLATIONS OF A MAGNETIC STAR

By

A. C. BANERJI* *and* V. K. GURTU†

[Received on 20th April, 1965]

ABSTRACT

Quasi-uniform radial oscillations of a magnetic star have been considered. The magnetic field has been derived from a potential of the form S_n/r^{n+1} where S_n is the surface harmonics of degree n. It has been shown that in a steady state when the ratio of specific heats γ is 4/3, quasi-uniform oscillations are possible in a star in which the density varies inversely as $(2n+6)$th power of the distance from the centre (excluding a small finite solid core round the centre). It has also been shown that if the magnetic potential is of the form $S_2' r^2$ then homogeneous spherical stars are also capable of quasi-uniform radial oscillation and that the magnetic field increases the frequency of oscillation.

INTRODUCTION

H. W. Babcock[1] discovered certain magnetic variable stars of which the best observed is HD 1252. He also put forward a suggestion that the fluctuations in the magnetic field are due to free oscillations of the star in presence of its field.[2] In the present paper quasi-uniform radial oscillations of a more or less spherical magnetic star have been considered. Chandrasekhar[3] has shown that a spherical sysmetry of configuration is, in general, incompatible with the presence of the fluid motions and magnetic fields. But he also mentions an exception. He has shown that for stability the ratio of specific heats γ must be greater than 4/3. He thus infers that for the existence of stable equilibrium, it is necessary that the total magnetic energy of a system does not exceed its negative gravitational potential energy. This would require that the magentic field H must be sufficiently small. It appears that this criterion is satisfied in the case of sun which has more or less spherical shape. Besides this Cowling[2] has also emphasized that far from a large internal magnetic field being necessary, the mechanical forces evoked by the oscillation are quite strong enough on their own to produce a sufficiently short period without any appreciable assistance from forces of electromagnetic origin. We shall see later on that the shape of the magnetic star does not remain perfectly spherical as it pulsates. Of course, there will not be much difference as the magnetic field is assumed to be sufficiently small. We can thus call our magnetic star as psuedo-spherical in shape having sufficiently small magnetic field.

It[4] is known that in certain cases *e.g.* when H is subject to

$$\text{Curl } \overline{H} = \alpha \overline{H} \text{ where } \alpha \text{ is a constant,}$$

the magnetic field $\overline{H}$ can be obtained in spherical polar coordinates. In the present case we shall derive the magentic field H from the potential. It may be mentioned here that the material present in the gaseous star is assumed to be non-megnetic. Now[5] we know that Ω, the potential of a magnetic system, in

*4-A, Beli Road, Allahabad.

†Department of Mathematics, University of Allahabad, Allahabad.

regions in which there is no magnetisation must be a solution of Laplace's equation, and must therefore be capable of expansion in the form

$$\Omega = \left(\frac{S_1}{r^2} + \frac{S_2}{r^3} + \dots\dots\right) + (S_0' + S_1'r + S_2'r^2 + \dots\dots) \qquad \dots\dots (1)$$

in which $S_1, S_2, \dots . S_0'\ S_1', S_2' \dots$ are surface harmonics of degree indicated by the subscripts.

Assuming that the magnetic field arises entirely from magnetism inside the system, we can see that $S'_0 = S'_1 = S'_2 \dots . = 0$ and we shall have then

$$\Omega = \frac{S_1}{r^2} + \frac{S_2}{r^3} + \dots \qquad \dots\dots (2)$$

As a simple case, we take Ω to be $\frac{S_n}{r^{n+1}}$ where S_n, surface harmonics of degree n, will satisfy the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\phi^2} + n(n+1)\,S_n = 0 \qquad \dots\dots (3)$$

and will clearly be certain function of θ and ϕ

Evidently, H will be given by

$$H = \left[\left(\frac{\partial\Omega}{\partial r}\right)^2 + \left(\frac{\partial\Omega}{r\partial\theta}\right)^2 + \left(\frac{\partial\Omega}{r\sin\theta\partial\phi}\right)^2\right]^{\frac{1}{2}}$$

or $$H = \frac{l}{r^{n+2}} \qquad \dots\dots (4)$$

where $$l = l_n\left[(n+1)^2 S_n^2 + \left(\frac{\partial S_n}{\partial\theta}\right)^2 + \left(\frac{\partial S_n}{\sin\theta\partial\phi}\right)^2\right]^{\frac{1}{2}} \qquad \dots\dots (5)$$

where l_n (a small constant) is introduced to ensure that the magnetic field assumed is taken to be small.

Having decided about the magnetic field we now proceed to formulate the equation of radial motion. Let

$$v_r = \frac{dr}{dt},\ v_\theta = r\frac{d\theta}{dt},\ v_\phi = r\sin\theta\left(\frac{d\phi}{dt}\right) \qquad \dots\dots (6)$$

denote the three components of velocity v at a point whose co-ordinates are (r, θ, ϕ) at any instant of time 't'. Then neglecting the square of v_θ and v_ϕ the equation of radial motion can be written as

$$\frac{d^2r}{dt^2} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \dots\dots (7)$$

where p, ρ and (F_r, F_θ, F_ϕ) denote pressure, density and three components of the external force F.

As the star is under the joint action of electromagnetic and mechanical forces, the external force F will, in general, include gravitational, magnetic and electrical forces. Thus F will consist of

(i) *Gravitational Force.*—Let it be noted by 'g' which will act along the radius vector, as the star is supposed to be more or less spherical in shape.

(ii) *Magnetic Force.*—The body force $\overline{F}_m$ of electromagnetic origin is given by

$$\overline{F}_m = \overline{j} \times \overline{B} \qquad \ldots . (8)$$

where $\overline{j}$ and $\overline{B}$ denote current density vector and magnetic induction vector.

For isotropic body which is not ferromagnetic, we have

$$\overline{B} = \mu \, \overline{H} \qquad \ldots . (9)$$

where μ is magnetic permeability.

Neglecting displacement current, one of the four fundamental equations of Maxwell's theory for bodies at rest can be written as

$$\text{Curl } \overline{H} = 4\pi \overline{j} \qquad \ldots . (10)$$

in electromagnetic units.

In view of (9) and (10) we get from (8)

$$\overline{F}_m = \frac{\mu}{4\pi} (\text{curl } \overline{H}) \times \overline{H} \qquad \ldots . (11)$$

In case the two vectors are equal we get a modified formula from Vector Analysis[7] *viz.*

$$\text{Curl } \overline{H} \times \overline{H} = -\tfrac{1}{2} \text{ grad } H^2 + (\overline{H}. \nabla) \overline{H} \qquad \ldots . (12)$$

and in view of this formula we get

$$\overline{F}_m = -\text{ grad} \left(\frac{\mu H^2}{8\pi}\right) + \frac{\mu}{4\pi} (\overline{H}. \nabla) \overline{H} \qquad \ldots . (13)$$

This equation implies[8] that the magnetic force $\overline{F}_m$ is equivalent to a hydrostatic pressure $\mu H^2/8\pi$ together with a tension $\mu H^2/4\pi$ along the lines of force. This is equivalent to a tension $\mu H^2/8\pi$ along the lines of force together with an equal pressure transverse to them. The physical interpretation of the second term $\frac{\mu}{4\pi} (\overline{H}. \nabla) \overline{H}$ is given in detail by Dungey.[1]

However, if, we assume that the lines of force are perpendicular to the radial direction, it can be seen that the tension $\mu H^2/8\pi$ will not contribute any force towards the radial direction (as tension acts along the lines force). Thus if p is the gas pressure the total pressure along the radial direction will become $(p + \mu H^2/8\pi)$.

(iii) *Electrical Force.*—As the conductivity is assumed to be infinite, the total electric field on the moving material will be zero.

Thus the equation (7) will become

$$\frac{d^2r}{dt^2} = -g - \frac{1}{\rho}\frac{\partial}{\partial r}\left(p + \frac{\mu H^2}{8\pi}\right) \qquad \ldots. (14)$$

We will now show that equation (14) is capable of representing quasi-uniform radial oscillations under certain conditions. But what are quasi-radial oscillations ?

In the case of pure uniform radial oscillations (which are possible in inviscous, non-rotating spherical non-magnetic star[9]) we know that[10]

(*i*) The distance between any two points is altered in the same way as the radius of the spherical star. Thus if R_0 and R be the radii of a spherical star before and after expansion (or contraction) and r_0 and r are the distances of any specified element of matter from the centre before and after expansion, then in the case of uniform expansion, we shall have

$$\frac{R}{R_0} = \frac{r}{r_0} \qquad \ldots. (15)$$

Putting $R = R_0(1 + R_1)$ and $r = r_0(1 + r_1)$ we find that $R_1 = r_1$ *i.e.* in the case of uniform expansion (or contraction) the amplitude $r_0 r_1$ at any point on a certain radius vector is the same fraction of the corresponding radial distance r_0. Thus for uniform expansion r_1, will be independent of r_0.

(*ii*) The frequency and the amplitude both are independent of θ and ϕ and therefore they remain the same for all points on a particular spherical shell concentric with the star. Thus condition (*i*) holds throughout the star.

But in the case of a star subject to small magnetic field, we shall see later on that the frequency of the so-called uniform radial oscillation obtained on applying condition (*i*) does not satisfy condition (*ii*) *i.e.*, the frequency and the amplitude both will depend upon θ and ϕ *i.e.* frequency will be different as we move along different radii vectors. For the above mentioned reason, the shape of the magnetic star will not remain perfectly spherical as it pulsates.

In view of the above facts we define such radial oscillations as "Quasi-Uniform". Evidently the surface of the magnetic star shall have undulatory character.

Let[11]

$$r = r_0(1 + r_1) \qquad \ldots. (16)$$

$$p = p_0(1 + p_1) \qquad \ldots. (17)$$

$$\rho = \rho_0(1 + \rho_1) \qquad \ldots. (18)$$

where r_1, ρ_1, p_1 are small quantities of first order. In the case of quasi-uniform oscillation r_1 is also independent of r_0. The letter with suffix zero represents the corresponding value in the undisturbed state.

Also,

$$g = \frac{GM(r)}{r^2} \qquad \ldots. (19)$$

where G is the constant of gravitation and $M(r)$ is the mass inside the sphere of radius r.

For adiabatic changes, pressure and density are connected by the relation

$$p = k\rho^{\gamma} \qquad \ldots . (20)$$

where γ is the effective ratio of specific heats (regarding the matter and enclosed radiation as one system) and k is a constant.

We shall assume here that as the star pulsates the mass inside the sphere of radius 'r_0' remains the same, so that

$$M(r) = M(r_0) \qquad \ldots . (21)$$

Furthermore, the star can be divided into similar and similarly situated thin spherical shells. As the star pulsates the mass of each such thin shell will also be conserved. Thus, we shall have

$$4\pi r^2 \, dr. \, \rho = 4\pi r_0^2 \, dr_0 \, \rho_0 \qquad \ldots . (22)$$

For the undisturbed state

$$g_0 = \frac{GM(r_0)}{r_0^2} \qquad \ldots . (23)$$

and $\quad p_0 = k\rho_0^{\gamma} \qquad \ldots . (24)$

From (16), (18) and (22) $\quad \rho_1 = -3r_1 \qquad \ldots . (25)$

From (17), (18), (20), (24) and (25) $p_1 = -3\gamma r_1 \qquad \ldots . (26)$

Therefore, from (16), (19), (21) and (25)

$$g = g_0 (1 - 2r_1) \qquad \ldots . (27)$$

From (17), (22), (26)

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \{ 1 - (3\gamma - 2) r_1 \} \frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0} \qquad \ldots . (28)$$

From (4), (16), (18) and (25)

$$\frac{1}{\rho}\frac{\partial}{\partial r}\left(\frac{\mu H^2}{8\pi}\right) = -\frac{(n+2)\mu l^2}{4\pi\rho_0 \, r_0^{2n+5}} \{1 - 2(n+1) r_1\} \qquad \ldots . (29)$$

Also, differentiating (16) with respect to 't', we get

$$\frac{d^2 r}{dt^2} = r_0 \frac{d^2 r_1}{dt^2} \qquad \ldots . (30)$$

In view of (27), (28), (29) and (30), equation (12) of motion becomes

$$r_0 \frac{d^2 r_1}{dt^2} = \left[\left\{ -g_0 - \frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0} + \frac{(n+2)\mu l^2}{4\pi\rho_0 \, r_0^{2n+5}} \right\} \right.$$

$$\left. + \left\{ 2g_0 + (3\gamma - 2)\frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0} - \frac{(n+1)(n+2)\mu l^2}{2\pi\rho_0 \, r_0^{2n+5}} \right\} \right] \ldots . (31)$$

which breaks up into the equation of magnetohydrostatic equilibrium

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0} = -g_0 + \frac{(n+2)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+5}} \qquad \ldots .(32)$$

and the equation of motion (after substituting the value of $\frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0}$ from (32))

$$\frac{d^2 r_1}{dt^2} = -\left\{ (3\gamma-4)\frac{g_0}{r_0} + \frac{(n+2)\,(2n+4-3\gamma)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+6}} \right\} \qquad \ldots .(33)$$

Evidently (33) represents an oscillatory motion and the frequency σ of the oscillation is given by

$$\sigma^2 = \left\{ (3\gamma-4)\frac{g_0}{r_0} + \frac{(n+2)\,(2n+4-3\gamma)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+6}} \right\} \qquad \ldots .(34)$$

We now define the mean density $\bar{\rho}_0$ of the sphere through (r_0, θ, ϕ) by the equation

$$\bar{\rho}_0 = \frac{3M(r_0)}{4\pi r_0^3} \qquad \ldots . (35)$$

In view of (23) and (35) equation (34) becomes

$$\sigma^2 = \left\{ \tfrac{4}{3}\pi\,(3\gamma-4)\;G\bar{\rho}_0 + \frac{(n+2)\,(2n+4-3\gamma)\mu l^2}{4\pi\rho_0\, r_0^{2n+6}} \right\} \qquad \ldots .(36)$$

We know that for stable quasi uniform oscillation σ^2 must be positive and independent of r_0.

For $\frac{4}{3} \leqslant \gamma < 2$ we see from (36) that σ^2 will remain positive but, in general, the expression on the right hand side of (36) will not be independent of r_0 *i.e.* in general it appears that quasi-uniform oscillations will not be possible. But there is one exception. Chandrasekhar[3] has mentioned that in a configuration for which $\gamma = \frac{4}{3}$, the total energy becomes zero and it remains in a steady state. He, therefore, concludes that a small radial expansion during which the configuration changes from one equilibrium state to an adjacent equilibrium state without any expenditure of energy is possible. Similarly a small contraction will also be possible. Hence there is every possibility that such a configuration, for which $\gamma = \frac{4}{3}$, may pulsate.

In the light of above information we can see that when $\gamma = \frac{4}{3}$ the right hand side of (36) can be made independent of r_0 and then the frequency for quasi-uniform oscillation will be given by

$$\sigma^2 = \frac{n(n+2)\mu l^2}{2\pi\beta} \qquad \ldots . (37)$$

if $$\rho_0\, r_0^{\,2n+6} = \beta \text{ (constant)} \qquad \ldots . (38)$$

Cowling[2] has remarked that pure magnetic oscillations, in Schwarzschild's sense (It is one in which the magnetic terms are more important than the mechanical terms) do not, in general, take place.

It is, therefore, interesting to note that "perfectly pure" magnetic oscillations (quasi-uniform) are possible in a more or less spherical megnetic star in which the density varies inversely as $(2n+6)$th power of the distance from the centre. To avoid singularity at the centre, we can exclude a small solid core of constant density around the centre of the star. The magnetic oscillations are described as "perfectly pure" because one can see from (37) that σ^2 is purely a function of μl^2 which is a magnetic term.

There is also one more case in which σ can be made independent of r_0.

If $n = -3$, *i.e.* $H = lr$ and $\overline{\rho_0} = \rho_0 = \rho_c$ (constant) *i.e.* uniform density, then from (36)

$$\sigma^2 = \left\{ \frac{4}{3}\pi(3\gamma-4)\, G\rho_c + \frac{(3\gamma+2)\,\mu l^2}{4\,\pi\,\rho_c} \right\} \qquad \ldots . (39)$$

In this case the physical picture is a little bit different. It is such that if a pseudo-spherical magnetic star (having uniform density) produces a magnetic field of the type $H = lr$ (which requires that the magnetic potential should be of the form $S_2'r^2$ where S_2' is a solution of (3) then it will be capable of quasi-uniform oscillation with frequency given by (39).

It is clear that in the absence of magnetic field σ will be given by

$$\sigma^2 = \tfrac{4}{3}\pi\,(3\gamma-4)\; G\rho_c \qquad \ldots . (40)$$

a result which was obtained by H. K. Sen.[9]

Comparing (39) and (40) we see that if a star, having uniform density, produces a magnetic field of the type $H = lr$ then its spherical form will be slightly distorted and it will then pulsate with a comparatively larger frequency.

As it has been pointed out earlier, we can see from (36), (37) or (39) that σ will differ as we move from one radius vector to another (for l is a function θ and ϕ both). Therefore the pulsating pieces of elementary masses on the surface of the star will have different frequencies and this will make the surface undulatory in character. But the difference in the frequencies along different radii vectors will be appreciably small because one can see from (36) that σ is a small quantity (as G and l are small quantities). Therefore the period of pulsation will be large and the star will pulsate quite slowly. Hence due to slow pulsation the star shall remain stable and the undulatory character of its surface will also be preserved.

In the end, the first author (A. C. B.) thanks C. S. I. R. (India) and the second author (V. K. G.) thanks U. G. C. for the award of research grants.

REFERENCES

1. Dungey, J. W. Cosmic Electrodynamics, Chap 5, Art 5, 7, (1958).
2. Cowling, T. G. and Roy, M. N. *Astro. Soc.*, Vol. 112, No. 5, (1952).
3. Chandrasekhar, S. Hydrodynamic and Hydrostatic stability, Chap. XIII, Art, 117-(3, (1961).
4. Deutsch, A. J. Handbuch Der Physik, (1958).
5. Jeans, Sir, J. Electricity and Magnetism, Chap. XI, Art. 454, (1943).
6. Routh, E. J. A Treatise on Dynamics of a Particle, Chap. VII, (1898).
7. Weatherburn, C. E. Advanced Vector Analysis, Chap. I, Art. 8.
8. Cowling, T. G. Magnetohydrodynamics, Chaps. I, II ; Art. 1·32, 2·2.
9. Sen, H. K. Uniform Radial Oscillations of a star, *Proc. Nat. Acad. Sci.* (India), Vol. XII (1942).
10. Chandrasekhar, S. An Introduction to the Study of Stellar Structure, Chap. 2, Art. 7, (1938).
11. Eddington, Sir, A. S. The Internal Constitution of the Stars, Chap. VIII, Art. 127.

RADIAL OSCILLATIONS OF MAGNETIC STAR

By

A. C. BANERJI* and V. K. GURTU†

[Received on 20th April, 1965]

ABSTRACT

Radial oscillations of a magnetic star have been shown to be stable for two laws of density viz. (i) Uniform density (ii) density varying inversely as an integral power (excluding 1, 3) of the distance from the centre. The magnetic field is assumed to be derived from a scalar potential of the form

$$\left(\frac{S_1}{r^2} + \frac{S_2}{r^3} + \dots + \frac{S_n}{r^{n+1}}\right), \text{ where } S_1, S_2, \dots, S_n$$

are surface harmonics. It has further been shown that the oscillations remain stable even if star pulated with relatively larger amplitude and that the stability of oscillation is independent of frequency.

INTRODUCTION

It is well known that stars, in general, have general magnetic fields. Wrubal[1] is of the opinion that stellar field may be relices from the time when the stars were formed out of interstellar matter. In 1958, Babcock[2] catalogued highly selected 338 magnetic stars out of which the best observed was HD 1252 which was known to be a spectrum variable with period about 9·3 days. With the discovery by H. W. Babcock of stars with strong magnetic field, Astrophysicists started making attempts to interpret their properties in terms of oscillations. It was Schwarzschild[3] who, for the first time, put forward the suggestion that the fluctuation in the field are due to free oscillation of the star in presence of its field but his analysis was not free from objections. He found that to get a nine-day period an internal field of order 10^6 gauses was necessary. Ferraro and Memory[4] also advanced their theory more or less along the same line.

It is known that if a fluid, in a stable state of magneto-hydrostatic equilibrium is perturbed slightly it will execute small oscillations about the equilibrium state. The simplest such oscillations were found by Alfvén for the case of a uniform field in an infinitely conducting inviscous liquid. We know that RR Lyrae, which is the prototype of a populous group of intrinsic variable stars characteristics of Baade's Population II, is a pulsating magnetic star.

In 1952, Cowling[5] while discussing the Oscillation Theory of magnetic variable star, showed that far from a large internal magnetic field being necessary in order to produce a nine-day oscillation in stars like HD 1252 the mechanical forces evoked by the oscillation are quite strong enough on their own to produce a sufficiently short period without any appreciable assistance from forces of electromagnetic origin. According to Cowling a magnetic field in a star takes a time of the Order of the age of the galaxy to decay and can hardly be created or reversed in a few days. The apparent reversal of the field must accordingly be due to the motions which alter the fields relation to the line of sight.

---

*4-A, Beli Road, Allahabad-2.

†Department of Mathematics, University of Allahabad, Allahabad.

Chandrasekhar, Prendergast, Woltjer and Fermi are some of the prominent workers in the field who have obtained interesting results. Chandrasekhar and Fermi[6] have discussed in great detail many problems of gravitational instability in the presence of magnetic field and their results have been extensively extended and improved by others. Prendergast[7] has found an equilibrium configuration in which the magnetic forces do not vanish. He has shown that an axisymmetric form of magnetic field is strictly compatible with the spherical boundaries in an incompressible medium. Woltjer[8] has extended this result for compressible medium.

In the present paper we shall be discussing the stability of radial oscillations of a spherical shaped magnetic star in which viscous forces are negligible. The magnetic field has been derived from a scalar potential[9] which satisfies Laplace's equation and hence can be expressed as

$$\Omega = \frac{S_1}{r^2} + \frac{S_2}{r^3} + \cdots\cdots + \frac{S_n}{r^{n+1}} \qquad \ldots. (1)$$

provided the magnetism arises entirely from inside the system. Here $S_1, S_2 \ldots S_n$ are the surface harmonics of degrees denoted by the suffixes.

As a simple case we take Ω to be S_n/r^{n+1} but it can be seen that the result will remain unaffected even if we take the whole series given by (1).

Clearly then, magnetic field H will be given by

$$H = \frac{l}{r^{n+2}} \qquad \ldots. (2)$$

where $l = \quad l_n \left[(n+1)^2 S_n^2 + \left(\frac{\partial S_n}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta} \frac{\partial S_n}{\partial \phi}\right)^2 \right]^{\frac{1}{2}} \qquad \ldots. (3)$

where l_n is a small constant, introduced to ensure that the magnetic field assumed is taken to be small. Clearly l will be a function of θ and ϕ (where (r, θ, ϕ) are spherical polar coordinates) and can therefore be treated as constant along a particular radius vector.

EQUATION OF RADIAL MOTION

Let $v_r = \frac{dr}{dt}$, $v_\theta = r \frac{d\theta}{dt}$, $v_\phi = r \sin\theta \frac{d\phi}{dt} \qquad \ldots. (4)$

denote the three components of the velocity of any fluid particle at time 't'. Neglecting the squares of v_θ and v_ϕ the equation[10] of radial motion can be written as

$$\frac{d^2r}{dt^2} = F_r - \frac{1}{\rho} \frac{\partial p}{\partial r} \qquad \ldots. (5)$$

where F_r is the component of external force F in the direction of radius vector 'r' and p and, ρ the pressure and density respectively.

Now, the external force F, in general, will include gravitational force, magnetic force and electrical force :

(i) *Gravitational Force*—Let it be denoted by 'g'.

(*ii*) *Magnetic Force.*—If $\bar{j}$ and $\bar{B}$ denote current density and magnetic induction, then the body force of electromagnetic origin will be $\bar{j} \times \bar{B}$ which in an isotropic medium will become $\bar{j} \times \mu\bar{H}$ where μ is magnetic permeability.

In view of the earlier works relating mainly to the general magnetic field of the sun or the interpretation of sunspots, by Cowling (origin and decay time of magnetic fields), Ferraro (interaction with rotation) Alfvén and Welén (magneto-hydrodynamic waves), we can neglect displacement currents and take the electrical conductivity to be infinite.

Using one of the Maxwell's equations (neglecting displacement currents) and some of the important formulae of vector analysis we can express magnetic force[11] as

$$-\operatorname{grad}\left(\frac{\mu H^2}{8\pi}\right) + \operatorname{div}\left(\frac{\mu \bar{H}\,\bar{H}}{4\pi}\right) \qquad \ldots . (6)$$

which shows that the magnetic force is equivalent to a hydrostatic pressure $\mu H^2/8\pi$ all round together with a tension $\mu H^2/4\pi$ along the lines of force. If we now assume that the lines of force are perpendicular to the radial direction, it can be seen that the tension $\mu H^2/4\pi$ will have no effect along the radius vector. Thus if p is the gas pressure the total pressure along the radius vector will be $p + \mu H^2/8\pi$.

(*iii*) *Electrical Force.*—This force will be almost zero as the conductivity is assumed to be infinite.

Thus the equation of radial motion will become

$$\frac{d^2r}{dt^2} = -g - \frac{1}{\rho}\frac{\partial}{\partial r}\left(p + \frac{\mu H^2}{8\pi}\right) \qquad \ldots . (7)$$

We shall deal with two types of oscillations *viz.* (*i*) Small oscillation, in which amplitude of oscillation will be a small quantity of first order and (*ii*) relatively large oscillation, in which amplitude of oscillation will be such that its squares will be retained while other higher order terms will be neglected.

Following Eddington, Sterne, Banerji, we put

$$r = r_0(1 + r_1) \qquad \ldots . (8)$$

$$p = p_0(1 + p_1) \qquad \ldots . (9)$$

$$\rho = \rho_0(1 + \rho_1) \qquad \ldots . (10)$$

where letters with suffix zero represent the corresponding values in the undisturbed state.

SMALL RADIAL OSCILLATIONS

It may be mentioned here that in the present case r_1, p_1, ρ_1 will all be small quantities of first order.

Now, $$g = GM(r)/r^2 \qquad \ldots . (11)$$

where G is the constant of gravitation and $M(r)$ the mass inside the sphere of radius 'r'.

For adiabatic changes p and ρ are connected by the relation $p = k\rho^{\gamma}$ . . . (12)

where γ is the effective ratio of specific heats (regarding the matter and enclosed radiations as one system).

We shall assume here that as the star pulsates, the mass inside the sphere of radius 'r' is the same as the mass inside the sphere of radius r_0, where r_0 is the value of r in the undisturbed state so that

$$M(r) = M(r_0) \qquad \ldots . (13)$$

Furthermore, the star can be divided into similar and similarly situated thin spherical shells. As the star pulsates the mass of each such thin shell shall be conserved. Thus, we shall have

$$4\pi\rho r^2 dr = 4\pi\rho_0 r_0^2\, dr_0 \qquad \ldots . (14)$$

For the undisturbed state

$$g_0 = \frac{GM(r_0)}{r_0^2} \qquad \ldots . (15)$$

and
$$p_0 = k\rho_0^{\gamma} \qquad \ldots . (16)$$

Now from (8), (10), (14) $\rho_1 = -3r_1 - r_0\, r_1'$ . . . . (17)

and from (9), 10), (12), (16), (17). $p_1 = -\gamma\,(3r_1 + r_0 r_1')$ . . . (18)

where
$$r_1' = \frac{dr_1}{dr_0}, \; r_1'' = \frac{d^2 r_1}{dr_0^2}$$

Therefore, from (8), (11), (13), (15), $g = g_0\,(1 - 2r_1)$ . . . . (19)

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \left[-\frac{p_0\gamma}{\rho_0}\,(4r_1' + r_0\, r_1'') + \{1 - (3\gamma - 2)\, r_1 - \gamma\, r_0\, r_1'\}\,\frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0}\right] \qquad \ldots . (20)$$

$$\frac{1}{\rho}\frac{\partial}{\partial r}\left(\frac{\mu H^2}{8\pi}\right) = -\frac{(n+2)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+5}}\,\{1 - 2\,(n+1)\, r_1 + r_0 r_1'\} \qquad \ldots . (21)$$

Differentiating (8) with respect to 't' we get

$$\frac{d^2 r}{dt^2} = r_0\,\frac{d^2 r_1}{dt^2} \qquad \ldots (22)$$

In view of (19), (20), (21), (22), equation of radial motion becomes

$$\frac{d^2 r_1}{dt^2} = \left\{ -\frac{g_0}{r_0} - \frac{1}{\rho_0\, r_0}\frac{\partial p_0}{\partial r_0} + \frac{(n+2)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+6}} \right\}$$

$$+ \left[\frac{2g_0}{r_0} r_1 + \frac{1}{\rho_0 r_0}\left\{ p_0\gamma\,(4r_1' + r_0 r_1'') + \left((3\gamma - 2)\, r_1 + \gamma r_0 r_1'\right)\frac{\partial p_0}{\partial r_0}\right\} \right.$$

$$\left. + \frac{(n+2)\,\mu l^2}{4\pi\rho_0 r_0^{2n+6}}\,\{r_0 r_1' - 2\,(n+1)\, r_1\} \right] \qquad \ldots . (23)$$

which breaks up into the equation of magnetohydrostatic equilibrium

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial r_0} = -g_0 + \frac{(n+2)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+5}} \qquad \ldots . \ (24)$$

and the equation for the deviation from the equilibrium (on using (24))

$$\frac{d^2 r_1}{dt^2} = \Bigg[-\left\{ (3\gamma-4)\frac{g_0}{r_0} + \frac{(n+2)(2n+4-3\gamma)\,\mu l^2}{4\pi\rho_0\, r_0^{2n+6}} \right\} r_1$$

$$+ \left\{ \gamma\left(\frac{4p_0}{\rho_0 r_0} - g_0\right) + \frac{(\gamma+1)(n+2)\,\mu l^2}{4\pi\rho_0 r_0^{2n+5}} \right\} r_1' + \frac{\gamma p_0\, r_1''}{\rho_0} \Bigg] \qquad \ldots . \ (25)$$

Assuming that (25) represents an oscillatory motion we put

$$r_1 = a_1 \cos \sigma t \qquad \ldots . \ (26)$$

in (25), to get

$$\frac{d^2 a_1}{dx^2} + \left\{ \frac{4}{x} - \frac{g_0\rho_0 R}{p_0} + \frac{(\gamma+1)(n+2)\,\mu l^2}{4\pi\gamma p_0\, R^{2n+4}\, x^{2n+5}} \right\} \frac{da_1}{dx}$$

$$+ \left\{ \frac{\sigma^2 R^2 \rho_0}{\gamma p_0} - \frac{(3\gamma-4)\, R g_0 \rho_0}{\gamma x p_0} - \frac{(n+2)(2n+4-3\gamma)\,\mu l^2}{4\pi\gamma\, p_0\, R^{2n+4}\, x^{2n+5}} \right\} a_1 = 0 \qquad \ldots . \ (27)$$

where σ is the frequency of oscillation, R is the radius of the star in the undisturbed state and x is such that $0 \leqslant x \leqslant 1$ and is connected with r_0 and R by $r_0 = Rx$.

Boundary Condition.—At the boundary $p_0 = 0$ and $x = 1$ and therefore (27) gives

$$[\,4\pi\gamma\, g_0'\, \rho_0'\, R^{2n+5} - (\gamma+1)(n+2)\,\mu l^2\,] \left(\frac{da_1}{dx}\right)_{x=1}$$

$$= [4\pi\rho_0'\sigma^2 R^{2n+6} - 4\pi(3\gamma-4)\, g_0'\, \rho_0'\, R^{2n+5} - (n+2)(2n+4-3\gamma)\mu l^2]\ (a_1)_{x=1} \qquad \ldots . \ (28)$$

provided $\left(\dfrac{d^2 a_1}{dx^2}\right)_{x=1}$ is finite.

Here g_0' and ρ_0' denote the values of g_0 and ρ_0 at $x = 1$.

It may be observed that equation (27) is a device for finding out the fraction 'a_1' of amplitude in terms of the fraction 'x' of the corresponding radial distance.

We shall now discuss the convergency of a_1, (which in turn will tell whether the oscillations are stable or not), for two laws of density.

Case 1.—Let us consider a sphere of uniform density $\bar{\rho}$. In this case, we can write in terms of x,

$$g_0 = \tfrac{4}{3}\pi\, G\bar{\rho} R x \qquad \ldots . \ (29)$$

and

$$p_0 = \tfrac{2}{3}\pi\, G\bar{\rho}^2 R^2 (1 - x^2) - \frac{\mu l^2}{8\pi R^{2n+4}} \left(\frac{1}{x^{2n+4}} - 1\right) \qquad \ldots . \ (30)$$

in which constant of integration is determined by the boundary condition, that is $p_0 = 0$ at $r_0 = R$.

Therefore (27) will become

$$x^2 (l_1 x^{2n+6} - l_2 x^{2n+4} + l_3) \frac{d^2 a_1}{dx^2}$$

$$+ x (l_4 x^{2n+6} - l_5 x^{2n+4} - l_6) \frac{da_1}{dx}$$

$$+ (l_7 x^{2n+6} + l_8) a_1 = 0 \quad \ldots . (31)$$

where
$$l_1 = 16\gamma G\pi^2\bar{\rho}^2 R^{2n+6},\ l_2 = \gamma (3\mu l^2 + 16 G\pi^2\bar{\rho}^2 R^{2n+6})$$
$$l_3 = 3\mu l^2\gamma,\ l_4 = 6l_1,\ l_5 = 4l_2$$
$$l_6 = (n\gamma + n + 2) 6\mu l^2$$
$$l_7 = 8\pi\bar{\rho} R^{2n+6} \{4\pi\bar{\rho} G(3\gamma - 4) - 3\sigma^2\}$$
$$l_8 = 6(n + 2)(2n + 4 - 3\gamma) \mu l^2$$

and the differential equation at the boundary can be obtained by putting $x = 1$ in (31).

(31) is a differential equation of second order having regular singularities[12] (since the indicial equation has two different roots) at $x = 0$ and $x = 1$ (since $l_1 - l_2 + l_3 = 0$).

We therefore, assume for the solution of (31) the series

$$a_1 = \sum_{\lambda=0}^{\infty} b_\lambda x^{q+\lambda} \quad \ldots . (32)$$

Putting (32) in (31), and equating to zero the coefficient of lowest degree term *viz.* x^q to zero, we get the indicial equation whose roots will be $2(n + 2)$ and $\{2(n + 2) - 3\}/\gamma$,

As (31) is a linear differential equation we may take $b_0 = 1$. We now expand the solution (32) about the origin and obtain the recurrence formula as

$$b_\lambda \{ (q + \lambda)(q + \lambda - 1) l_1 + (q + \lambda) l_4 + l_7 \}$$
$$- b_{\lambda+2} \{ (q + \lambda + 2)(q + \lambda + 1) l_2 + (q + \lambda + 2) l_5 \}$$
$$+ b_{\lambda+2n+6} \{ (q + \lambda + 2n + 6)(q + \lambda + 2n + 5) l_3$$
$$- (q + \lambda + 2n + 6) l_6 + l_8 \} = 0 \quad \ldots . (33)$$

This on arranging in powers of λ, dividing by b_λ and then taking limit as $\lambda \to \infty$, gives

$$l_1 - l_2 A^2 + l_3 A^{2+n+6} = 0 \quad \ldots . (34)$$

where

$$\underset{\lambda \to \infty}{Lt} \frac{b_{\lambda+1}}{b_\lambda} = A = 1 \quad \ldots . (35)$$

as (34) is satisfied by $A = \pm 1$ the negative sign being neglected as it makes (32) divergent.

Hence the series solution for (31) has unit radius of convergence. As (31) has regular singularitis at $x = 0$ and $x = 1$, the series solution (32) will be convergent in the neighbourhood of the origin and will remain convergent right upto next singularity $x = 1$.

We now therefore, test the convergence of (32) for $x = 1$.

In the view of (35), all the terms will ultimately be of the same sign and we have

$$\frac{b_{\lambda+1}}{b_{\lambda}} = 1 - \varepsilon \qquad \ldots . (36)$$

where $\varepsilon = O\left(\frac{1}{\lambda^{s}}\right)$, S, being a positive integer.

Therefore the recurrance formula (33) will, on making use of (36) (treating ε to be first order quantity) and taking the limit as $\lambda \to \infty$, give

$$\varepsilon = \frac{L}{\lambda} \qquad \ldots . (37)$$

where

$$L = \left[\frac{16G\pi^{2}\bar{\rho}^{2}R^{2n+6} - 3\mu l^{2}(n+2)\left(1 - \frac{1}{\gamma}\right)}{16G\pi^{2}\bar{\rho}^{2}R^{2n+6} - 3\mu l^{2}(n+2)}\right] \qquad \ldots . (38)$$

Hence, we have more clearly

$$\frac{b_{\lambda+1}}{b_{\lambda}} = 1 - \frac{L}{\lambda} + O\left(\frac{1}{\lambda^{2}}\right) \qquad \ldots . (39)$$

which shows that the series solution (32) is also convergent for $x = 1$ since $L > 1$ (Gauss's rule)[13].

As a_1 remains finite at $x = 1$, it is possible that the boundary condition shall be satisfied.

Thus if the magnetic star having uniform density sets oscillating, the small oscillation will remain stable *i.e.* the star will continue to oscillate.

In the absence of magnetic field, L becomes unity and then in view of (39) series solution (32) will become divergent at the boundary $x = 1$. Hence, in general, the radial oscillations of homogeneous spherical star will be unstable in the absence of magnetic field. However, Sterne[14], by making the series solution terminate with some terms, has been able to find out certain radial modes for such stars.

It may be noted here that the very presence of magnetic term in L makes it greater than unity and thus enables a_1 to remain finite at $x = 1$. This clearly proves that the magnetic field is responsible for making star stable for radial oscillation. In other words, the magnetic field increases the stability of radial oscillations.

In the opinion of Kopal[15], δ Cephei $-F_5$ stars seem to approach the limit of homogeneity. Chandrasekhar[16] states "the giants should be expected to be more homogeneous than the main series stars". Eddington has also emphasized that the Cepheids should necessarily be more homogeneous than ordinary stars.

In the light of above information one can easily deduce that in the presence of magnetic field of the form given in (2), the Cepheid Variable stars will remain stable as they pulsate.

Case II.—Variable Density.

We small now consider the stability of small radial oscillation of Banerji's model[17] with a small, homogeneous central core and the density in the annulus varying inversely as an integral power of the distance from the centre.

Thus, let $$\rho_0 = \frac{K}{r_0^{m}} \qquad \ldots . (40)$$

where m is any positive integer (excluding 1 and 3 as for these values, g_0 and p_0 become indeterminate) and K, a constant, is given by

$$K = \frac{(m-3)\bar{\rho}\,(1-\beta^3)\,\beta^{m-3}\,R^m}{3(1-\beta^{m-3})}$$

$$\text{where } \beta = \frac{\text{radius of central core}}{\text{radius of star in undisturbed state } (R)}$$

$$\bar{\rho} = \text{mean density in the annulus}$$

As in the previous case we can obtain in terms of x,

$$\rho_0 = \frac{K}{R^m\,x^m} \qquad \ldots . (42)$$

$$g_0 = \frac{4\pi KG}{3(m-3)\,x^2\,R^{m-1}}\left(\frac{m}{\beta^{m-3}} - \frac{3}{x^{m-3}}\right) \qquad \ldots . (43)$$

$$p_0 = \left[\frac{4\pi GK^2}{3(m-3)\,R^{2m-2}}\left\{\frac{m}{(m+1)\,\beta^{m-3}}\left(\frac{1}{x^{m+1}} - 1\right) - \frac{3}{2(m-1)}\left(\frac{1}{x^{2m-2}} - 1\right)\right\} - \frac{\mu l^2}{8\pi R^{2n+4}}\left(\frac{1}{x^{2n+4}} - 1\right)\right] \qquad \ldots . (44)$$

Putting these values in (27), we get

$$x^2(m_1 x^{2n+m-1} - m_2\,x^{2n+2} - m_3\,x^{2m-4} + m_4\,x^{2m+2n})\frac{d^2a_1}{dx^2}$$

$$- x\,(m_5\,x^{2n+m-1} - m_6\,x^{2n+2} - m_7\,x^{2m-4} - m_8\,x^{2m+2n})\frac{da_1}{dx}$$

$$- (m_9\,x^{2n+m-1} - m_{10}\,x^{2n+2} + m_{11}\,x^{2m-4} - m_{12}\,x^{m+2n+2})a_1 = 0 \qquad \ldots . (45)$$

where

$$
\left.
\begin{aligned}
m_1 &= \frac{4\pi m\, GK^2}{3(m-3)(m+1)\,\beta^{m-3}\,R^{2m-2}} \\
m_2 &= \frac{2\pi GK^2}{(m-1)(m-3)\,R^{2m-2}},\quad m_3 = \frac{\mu l^2}{8\pi R^{2n+4}} \\
m_4 &= \left[\frac{4\pi GK^2}{3(m-3)R^{2m-2}}\left\{\frac{3}{2(m-1)} - \frac{m}{(m+1)\,\beta^{m-3}}\right\}\right. \\
&\qquad\qquad \left. + \frac{\mu l^2}{8\,\pi\, R^{2n+4}}\right] \\
m_5 &= (m-3)\, m_1,\quad m_6 = 2(m-3)\, m_2 \\
m_7 &= \frac{(n\gamma+n+2)\mu l^2}{4\,\pi\,\gamma\, R^{2n+4}},\quad m_8 = 4m_1 \\
m_9 &= \frac{4m\,\pi\, GK^2\,(3\gamma-4)}{3(m-3)\,\gamma\,\beta^{m-3}\,R^{2m-2}} \\
m_{10} &= 4\,\pi\, G\,(3\gamma-4)\,K^2/\gamma(m-3)\,R^{2m-2} \\
m_{11} &= (n+2)\,(2n+4-3\gamma)\,\mu l^2/4\,\pi\,\gamma\, R^{2n+4} \\
m_{12} &= K\sigma^2/\gamma R^{2m-2}
\end{aligned}
\right\} \qquad (46)
$$

Equation to be satisfied at the boundary can be obtained by putting $x = 1$ in (45).

As in the previous case, equation (45) is a differential equation of second order having regular singularities at $x = 0$ and $x = 1$.

We, therefore, assume for the solution of (45), the series

$$a_1 = \sum_{\lambda=0}^{\infty} b_\lambda x^{q+\lambda} \qquad \ldots . (47)$$

and obtain the roots of indicial equation as $(2n+4)$ and $\left\{\frac{2(n+2)}{\gamma} - 3\right\}$. The recurrence formula, on dividing by λ^2 and taking the limit as $\lambda \to \infty$ will give

$$m_4 - m_2\, A^{2m-2} + m_1\, A^{\ \ +1} - m_3\, A^{2n+4} = 0 \qquad \ldots . (48)$$

where

$$A = \underset{\lambda\to\infty}{Lt}\ \frac{b_{\lambda+1}}{b_\lambda} \qquad \ldots . (49)$$

clearly $A = 1$ satisfies (48) irrespective of whether m is even or odd ; $A = -1$ will also satisfy it when m is odd. However, we shall take $A = 1$ only for the reason mentioned earlier and this in view of (49) gives

$$\underset{\lambda\to\infty}{Lt}\ \frac{b_{\lambda+1}}{b_\lambda} = 1 \qquad \ldots . (50)$$

showing thereby, that (47) has a unit radius of convergence. Hence the series solution (47) will be convergent in the neighbourhood of the centre and will remain so right upto $x = 1$. In order to study the convergence at $x = 1$, we can, by following the same method as used previously, obtain

$$\frac{b_{\lambda+1}}{b_{\lambda}} = 1 - \frac{S}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \qquad \ldots . \; (51)$$

where

$$S = \left(\begin{array}{l} \left\{ \dfrac{4\pi G K^2}{(m-3) R^{2m-2}} \left(\dfrac{m}{3\beta^{m-3}} - 1 \right) \right\} - \dfrac{(n+2) \left(1 - \dfrac{1}{\gamma} \right) \mu l^2}{4\pi R^{2n+4}} \\ \left\{ \dfrac{4\pi G K^2}{(m-3) R^{2m-2}} \left(\dfrac{m}{3\beta^{m-3}} - 1 \right) \right\} - \dfrac{(n+2) \mu l^2}{4\pi R^{2n+4}} \end{array} \right. \qquad \ldots . \; (52)$$

which is greater than 1, thus showing that the solution is convergent at $x = 1$ also.

Therefore, the model in question *viz.* Banerji's model is also capable of stable radial oscillation in presence of magnetic field. Banerji's model[17] was given by Professor A. C. Banerji (while discussing with the problem of the origin of the Planetary System in 1942) and that it was unstable for small oscillation was shown by H. K. Sen. The important point to note is the expression for S (and also that of L in the first case) from which it will be clear that it is only and only due to magnetic term that S or L becomes greater than unity; on which depends upon the covergence of the series solution. In other words, magnetic field increases the stabiliy of oscillation.

It is interesting to note that the second case includes the first case when $m = 0$ and $\beta \rightarrow 0$. This means that Cepheid Variables which are pulsating stars, are particular cases of Banerji's model.

It is further interesting to note that the stability of radial oscillations, for both the laws of density, does not depend upon the frequency (as the convergence of the series solution corresponding to a_1 depends upon L and S which are independent of σ) and therefore also not on the period P of pulsation. Therefore, we arrive at an important conclusion that the small radial oscillations will remain stable however small or large the period of pulsation may be.

Relatively—Larger Radial Oscillation.

In this case, equation (24) of magneto-hydrostatic equilibrium will remain the same while equation (25) for the deviation from the equilibrium will be modified as

$$\begin{aligned} \frac{d^2 r_1}{dt^2} = \Bigg[& \frac{g_0}{r_0} \Big\{ (4-3\gamma) r_1 + \left(\frac{9\gamma^2 - 9\gamma - 4}{2} \right) r_1^2 - \gamma r_0 r_1' + \gamma (3\gamma - 1) r_0 r_1 r_1' \\ & \qquad + \gamma r_0 r_1'^2 + \tfrac{1}{2} \gamma (\gamma - 1) r_0^2 r_1'^2 \Big\} \\ & + \frac{p_0}{\rho_0 r_0} \Big\{ 4\gamma r_1' - 4\gamma (3\gamma - 1) r_1 r_1' - \gamma r_1'^2 - 4\gamma^2 r_0 r_1'^2 + \gamma r_0 r_1'' \\ & \qquad - \gamma (3\gamma - 1) r_0 r_1 r_1'' - 2\gamma r_0 r_1' r_1'' - \gamma(\gamma - 1) r_1' r_1'' r_0^2 \Big\} \\ & - \frac{(n+2)\mu l^2}{4\pi \rho_0 r_0^{2n+6}} \left\{ \begin{array}{l} (2n+4-3\gamma) r_1 + \tfrac{1}{2} (9\gamma^2 - 9\gamma - 4n^2 - 10n - 4) r_1^2 - (\gamma + 1) r_0 r_1' \\ + (3\gamma^2 + 2n - \gamma + 3) r_0 r_1 r_1' + \gamma r_0 r_1'^2 + \tfrac{1}{2} \gamma (\gamma - 1) r_0^2 r_1'^2 \end{array} \right\} \Bigg] \end{aligned} \qquad \ldots . \; (53)$$

Assuming that (53) represents an oscillatory motion we put in it

$$r_1 = a_1(r_0)\cos\sigma t - a_2(r_0)\cos 2\sigma t - a_3(r_0) \quad \ldots. (54)$$

(where $a_1(r_0)$ is such a quantity that its square is retained and other higher powers are neglected and $a_2(r_0)$ and $a_3(r_0)$ are such small quantities that their squares and other higher order terms are neglected) and equate to zero the coefficients of $\cos\sigma t$, $\cos 2\sigma t$ and the terms independent of time, to get

$$\frac{d^2a_1}{dx^2} + \frac{da_1}{dx}\left\{\frac{4}{x} - \frac{g_0\rho_0 R}{p_0} + \frac{(\gamma+1)(n+2)\mu l^2}{4\pi\gamma p_0 R^{2n+4}x^{2n+5}}\right\}$$
$$+ \left\{\frac{\sigma^2R^2\rho_0}{\gamma p_0} - \frac{(3\gamma-4)Rg_0\rho_0}{\gamma x p_0} - \frac{(n+2)(2n+4-3\gamma)\mu l^2}{4\pi\gamma p_0 R^{2n+4}x^{2n+5}}\right\} a_1 = 0 \quad \ldots. (55)$$

$$\frac{d^2a_2}{dx^2} + \frac{da_2}{dx}\left\{\frac{4}{x} - \frac{g_0\rho_0 R}{p_0} + \frac{(\gamma+1)(n+2)\mu l^2}{4\pi\gamma p_0 R^{2n+4}x^{2n+5}}\right\}$$
$$+ \left\{\frac{4\sigma^2R^2\rho_0}{\gamma p_0} - \frac{(3\gamma-4)Rg_0\rho_0}{\gamma x p_0} - \frac{(n+2)(2n+4-3\gamma)\mu l^2}{4\pi\gamma p_0 R^{2n+4}x^{2n+5}}\right\} a_2 = Q_0 \quad \ldots. (56)$$

and $$\frac{d^2a_3}{dx^2} + \left\{\frac{4}{x} - \frac{g_0\rho_0 R}{p_0} + \frac{(\gamma+1)(n+2)\mu l^2}{4\pi\gamma p_0 R^{2n+4}x^{2n+5}}\right\}\frac{da_3}{dx}$$
$$- \left\{\frac{(3\gamma-4)Rg_0\rho_0}{\gamma x p_0} + \frac{(n+2)(2n+4-3\gamma)\mu l^2}{4\pi\gamma p_0 R^{2n+4}x^{2n+5}}\right\} a_3 = Q_0 \quad \ldots. (57)$$

where $$Q_0 = \Bigg[\left\{\frac{g_0}{Rx}\left(\frac{9\gamma^2 \; 9\gamma-4}{4}\right) + \frac{(n+2)(4n^2-9\gamma^2+9\gamma+10n+4)\mu l^2}{16\pi\rho_0 R^{2n+6}x^{2n+6}}\right\}a_1^2$$
$$+ \left\{\tfrac{1}{2}\gamma(3\gamma-1)g_0 - \frac{2\gamma(3\gamma-1)p_0}{\rho_0 Rx} - \frac{(n+2)(3\gamma^2+2n-\gamma+3)}{8\pi\rho_0 R^{2n+5}x^{2n+5}}\right\}a_1a_1'$$
$$+ \left\{\frac{\gamma g_0}{2} + \tfrac{1}{4}\gamma(\gamma-1)g_0Rx - \frac{\gamma p_0}{2\rho_0 Rx} - \frac{2p_0\gamma^2}{\rho_0} - \frac{\gamma(n+2)\mu l^2}{8\pi\rho_0 R^{2n+5}x^{2n+5}}\right.$$
$$\left. - \frac{\gamma(\gamma-1)(n+2)\mu l^2}{16\pi\rho_0 R^{2n+4}x^{2n+4}}\right\}a_1'^2 - \frac{1}{2\rho_0}\gamma(3\gamma-1)p_0a_1a_1''$$
$$- \frac{\gamma p_0}{\rho_0}\left\{1 + \tfrac{1}{2}(\gamma-1)Rx\right\}a_1'a_1''\Bigg] \quad \ldots. (58)$$

Boundary conditions.—As the boundary $x=1$ and $p_0=0$ and so (55), (56) and (57) give

$$\{4\pi\gamma g_0'\rho_0' R^{2n+5} - (\gamma+1)(n+2)\mu l^2\}\left(\frac{da_1}{dx}\right)_{n=1}$$
$$= \{4\pi\rho_0'\sigma^2R^{2n+6} - 4\pi(3\gamma-4)g_0'\rho_0'R^{2n+5} - (n+2)(2n+4-3\gamma)\mu l^2\} \times (a_1)_{x=1} = 0. \quad \ldots. (59)$$

$$\{4\pi\gamma g_0'\rho_0'R^{2n+5} - (\gamma+1)(n+2)\mu l^2\}\left(\frac{da_2}{dx}\right)_{x=1}$$
$$=\{16\pi\rho_0'\sigma^2R^{2n+6} - 4\pi(3\gamma-4)g_0'\rho_0'R^{2n+5} - (n+2)(2n+4-3\gamma)\mu l^2\} \times (a_2)_{x=1}+Q_0' \quad \ldots. (60)$$

$$\{4\pi\gamma g_0'\rho_0' R^{2n+5} - (\gamma+1)(n+2)\mu l^2\}\left(\frac{da_3}{dx}\right)_{x=1}$$
$$+ \{4\pi(3\gamma-4)g_0'\rho_0' R^{2n+5} + (n+2)(2n+4-3\gamma)\mu l^2\}(a_3)_{x=1}=Q_0' \quad \ldots. (61)$$

provided $\left(\frac{d^2a_1}{dx^2}\right)$, $\left(\frac{d^2a_2}{dx^2}\right)$ and $\left(\frac{d^2a_3}{dx^2}\right)$ all remain finite at $x=1$. Here g_0' ρ_0', Q_0' are the values of g_0, ρ_0 and Q_0 at $x=1$.

In order to study the stability of relatively large radial oscillation we shall consider the convergence of a_1, a_2, a_3 which are given by (55), (56) and (57) for two laws of density as in the case of small radial oscillations.

It is interesting to note that equation (55) is exactly the same as (27) which has already been proved to be convergent for both the laws of density. Hence the series corresponding to a_1, as given in (54), is convergent.

Secondly, the complete solution of (56) will consist of complimentary function which is the solution of an equation obtained on putting $Q_0 = o$ in (56) and the particular intergral which will be obtained by certain operation over Q_0 (which is a function of a_1). Now, since the convergence of (55) does not depend upon σ and the equation obtained on putting $Q_0 = o$ in (56) is the same as (55) except with the difference that in place of σ there is 2σ, we can easily conclude that complimentary function of a_2 will be convergent. The particular integral will also be convergent as Q_0 is a function of a_1, the series solution of which is convergent.

Similarly we can show that series solution corresponding to a_3 will also be convergent.

Hence the fraction r_1, given by (54), of amplitude of oscillation will remain finite and there is therefore every possibility that boundary conditions given by (59), (60) and (61) may also exist.

Thus, the relatively large radial oscillation will also remain stable for both the laws of density. It appears that the oscillations will remain stable for pretty large value of amplitude.

It view of the results obtained above we arrive at an important conclusion that if, in a star, the magnetism arises entirely from inside and the magnetic field is derived from a scalar potential of the form Sn/r^{n+1}, then the star will execute stable radial oscillation whatever the frequency may be. Chandrasekhar has also pointed out the possibility of the period of pulsation being arbitrarily long. Futhermore the stability of oscillations will be preserved even if the star pulsated with sufficiently large amplitude.

If, instead of taking only general term Sn/r^{n+1} for the magnetic potential, we take the whole series (1), it can be seen that the result will more or less remain the same in nature.

In the end, the first auther (A. C. B.) thanks C. S. I. R. (India) and the second author (V. K. G.) thanks U. G. C. for the award of research grants.

REFERENCES

1. Wrubel, M. H. *Ap. J.* **116** : 291, (1952).
2. Babcock, H. W. *Ap. J. Suppl.* 3, 141, (1958).
3. Schwarzschild, M. *Ann. d'Astrophys*, **12** : 148, (1949).
4. Ferraro, V. C. A. and Memory, D. J. *M. N. Roy. Astro. Soc.* **112** : 361, (1952).
5. Cowling, T. G. *M. N. Roy. Astro. Soc.*, **112** : 525, (1952).
6. Chandrasekhar, S and Fermi, E. *Ap. J.*, **118** : 116, (1953).
7. Prendergast, K. H. *Ab J.*, **123** : 498, (1956).
8. Woltjer, L. *Ap. J.*, **131** : 227, (1960).
9. Jeans, Sir J. "The Mathematical Theory of Electricity and Magnetism," Chap. XI.
10. Routh, E. J. "A Treatise on Dynamics of a Particle" Chapter, VII.
11. Cowling, T. G. "Magnetohydrodynamics". Chaps. 1, 2 ; Articles 1.32 and 2.2.
12. Whittaker, E. T. and Watson, G. N. "A course of Modern Analysis", Chap. X.
13. Bromwich, T. J "An Introduction to the Theory of Infinite Series", Chap. II.
14. Sterne, T. E. *M. N. Roy. Astro. Soc.*, **97** : 583, (1937).
15. Kopal, J. *M. N. Roy. Astro. Soc.*, **99** : 38, (1938).
16. Chandrasekhar, S. *M. N. Roy. Astro. Soc.*, **96**, 656, (1936).
17. Banerji, A. C. *Proc. Nat. Acad. Sci.* (India), **8** : 173, (1942).

ON SCHRODINGER'S EQUATION

By

R. L. DASVARMA

Department of Mathematics, L. S. College, Muzaffarpur

[Received on 14th December, 1965]

ABSTRACT

Schrodinger's equation, in quantum mechanics, is the postulate regarding the gradual and causal transformation of the state of an isolated system, as distinguished from the sudden, acausal change brought about by the interaction of the system with an external agency. In this respect, it is the quantum mechanical analogue of the equation of motion in classical mechanics. The validity of Schrodinger's equation in different cases has been examined here and it has been shown that the equation results from the complementary between time and energy and from the unitary nature of the transformation from one representation to another.

1. Quantum mechanics prescribes two distinct types of change in the state of a physical system governed by two entirely different processes. One is the sudden, irreversible and acausal change brought about by an external agency, such as a measuring apparatus and the other is the gradual, reversible and causal transformation of the state of an isolated system, described by a differential equation. The differential equation postulated for the description of the second type of process is Schrodinger's equation

$$\frac{ih}{2\pi}\frac{\partial\psi}{\partial t} = H\psi, \qquad (1\cdot1)$$

where $\psi\,(q, t)$ is the function of position, represented by the coordinates q, and of time t, and represents the state of the system, H is the Hamiltonian operator, representing the total energy of the system and h is Planck's constant.

The equation (1·1) is of the first order in t and enables us to determine $\psi\,(q, t)$ at any time t, provided it is given at the initial time t_0 In other words (1·1) establishes a relation between $\psi\,(q, t_0)$ and $\psi\,(q, t)$.

In this respect, the role of (1·1) is identical with that of the equation of motion in classical mechanics. Keeping in mind the difference in the definitions of "state" in the two theories, we note that both the equations connect causally the state at a later time with that at an earlier time.

2. We shall examine the validity of (1·1) in different cases. First, we assume that the initial state is a characteristic state of some observable, which is a constant of the motion. In this case, both $\psi\,(q, t_0)$ and $\psi\,(q, t)$ represent the same state and the equation (1·1) merely connects two representations of the same state. The position representation at time t is different from that at time t_0 and the transformation from one to the other is unitary. The unitary operator effecting this transformation will depend on t, or, more precisely, on the difference $t - t_0$, since there is no absolute zero-point for time.

In the case of a system with one degree of freedom, such as a particle moving along a straight line, the single coordinate at a given instant of time

defines a state. Let the system be in a state characteristic of the observable A, given by $A = \alpha$, and let A be a constant of the motion. The function $\psi(q, t; \alpha)$ determines the probability that the coordinate of time t is q, the observable A having the value α. If the coordinate and time can be determined simultaneously without mutual interference, they are stochastically independent and we may write

$$\psi(q\, t;\, \alpha) = u(q;\, \alpha)\, v(t;\, \alpha). \tag{2·1}$$

If the observable A is the energy E of the system, we have, on account of the complementarity between time and energy[1],

$$v(t;\, E) = \exp(-iEt/\hbar),\ \hbar = h/2\pi, \tag{2·2}$$

so that an energy-characteristic state is represented by

$$\psi(q, t;\, E) = u(q;\, E)\exp(-iEt/\hbar). \tag{2·3}$$

Denoting by $U(t, t_0)$, the unitary operator connecting the representations at t_0 and at t, we may write

$$\psi(q, t;\, E) = U(t, t_0)\,[\psi(q, t_0;\, E)]$$

i.e.,

$$u(q;\, E)\exp(-iEt/\hbar) = U(t, t_0)\,[\,u(q;\, E)\exp(-iEt_0/\hbar)]$$

or,

$$U(t, t_0)\,[u(q;\, E)] = \exp\{-iE(t - t_0)/\hbar\}\, u(q;\, E). \tag{2·4}$$

Now $u(q;\, E)$ is an eigenfunction of the Hamiltonian H, which is a Hermitian operator, the corresponding eigenvalue being E, so that

$$Hu(q;\, E) = Eu(q;\, E). \tag{2·5}$$

Equation (2·4) shows that $u(q;\, E)$ is an eigenfunction of $U(t, t_0)$ belonging to the eigenvalue $\exp\{-iE(t - t_0)\hbar\}$, which is of modulus unity.

Now, the function

$$f(x) = \exp\{-i(t - t_0)x/\hbar\}$$

transforms the real eigenvalues of the Hermitian operator H into complex eigenvalues of unit moduli of the unitary operator $U(t, t_0)$. Hence[2] the unitary operator $U(t, t_0)$ must be the function

$$f(H) = \exp\{-i(t - t_0)H/\hbar\}$$

of the Hermitian operator H, so that

$$U(t, t_0) = \exp\{-i(t - t_0)H/\hbar\}. \tag{2·6}$$

Incidentally, this also establishes that $U(t, t_0)$ depends on the defference $t - t_0$.

If the initial state is characteristic of some other observable A, which is a constant of the motion, it remains unchanged in time. Expressing the state $A = \alpha$ as a superposition of the energy-characteristic states, we can show that

$$\psi(q,t;\, \alpha) = U(t, t_0)\,[\psi(q, t_0;\, \alpha)], \tag{2·7}$$

where $U(t, t_0)$ is the unitary operator given by (2·6).

3. Since time and energy form a complementary pair of observables, the kernel of the Hamiltonian operator in time – representation is[3]

$$H(t) = \Sigma_j \; E_j \; \rho \; (t, t' \; ; E_j \;) = \Sigma_j \; E_j \; \psi \; (t \; ; E_j \;) \; \psi^* \; (t' \; ; E_j \;)$$
$$(1/2\pi\hbar)\Sigma_j \; E_j \; \exp \; \{iE_j \; (t' - t)/\hbar\}, \tag{3·1}$$

where $\rho \; (t, t' \; ; Ej \;)$ is the statistical density operator for the state $E = Ej$ in the time-representation.

If $f\;(t)$ is any function of t, we have

$$H\;(t)\;[f\;(t)\;] = (1/2\pi\hbar)\int_{-\infty}^{\infty} [\Sigma j \; Ej \exp \; \{iEj\;(t' - t)/\hbar\}]\; f\;(t')dt'$$
$$(1/2\pi\hbar)\;\Sigma j\; Ej \exp\;(-iEj\;t/\hbar)\int_{-\infty}^{\infty} f\;(t')\exp\;(+iEj\;t'/\hbar)\;;dt'$$
$$(1/2\pi\hbar,^{\frac{1}{2}}\;\Sigma j\;Ej\;gj\exp\;(-iEj\;t/\hbar),$$

where

$$g_j = (1/2\pi\hbar)^{\frac{1}{2}}\int_{-\infty}^{\infty} f\;(t)\exp\;(+iEj\;t/\hbar)\;dt.$$

Now,

$$\int_{-\infty}^{\infty}\frac{\partial f}{\partial t}\exp\;(iEj\;t/\hbar)\;dt = -\int_{-\infty}^{\infty} f\;(t)\;\frac{\partial}{\partial t}\;[\exp\;(iEj\;t/\hbar)]\;dt$$
$$= (-iEj\;/\hbar\;)\int_{-\infty}^{\infty} f\;(t)\exp\;(iEj\;t/\hbar)\;dt\; - iEj\;\;gj\;/\hbar.$$

Hence the Fourier expansion of $-(\hbar/i)\;\dfrac{\partial f}{\partial t}$ is the series

$$(1/2\pi\hbar)^{\frac{1}{2}}\;\Sigma_j\;Ej\;gj\exp\;(-iEj\;t/\hbar).$$

Thus, $H\;(t)\;f\;(t)$ and $-\;(\hbar/i)\;\dfrac{\partial f}{\partial t}$ have the same Fourier series for all function $f\;(t)$ of t. Therefore, in the time-representation, the Hamiltonian, operator has the form

$$H\;(t) = -(\hbar/i)\frac{\partial}{\partial t}. \tag{3·2}$$

Since $H\;(t)$ represents the Hamiltonian in the time-representation, the unitary operator $U\;(t,t_0)$ transforms $H\;(t_0)$ into $H\;(t)$, so that

$$U\;(t, t_0)\;H\;(t_0)\;U^{-1}\;(t, t_0) = H\;(t)\; = -\;(\hbar/i)\;\frac{\partial}{\partial t}\;,$$

or,
$$U\;(t, t_0)\;H\;(t_0)\; = -\;(\hbar/i)\frac{\partial}{\partial t}U(t, t_0).$$

But $U\;(t,\;t_0)$, being a function of $H\;(t_0)$, must commute with it.

Therefore,

$$-\;(\hbar/i)\;\frac{\partial}{\partial t}U\;(t,\;t_0) = H\;(t_0)\;U\;(t,\;t_0). \tag{3·3}$$

From (3·3) we have

$$i\hbar \frac{\partial}{\partial t} U\,(t, t_0)\,[\,\psi\,(q, t_0)] = H\,(t_0)\; U\,(t, t_0)\,[\psi\,(q, t_0)]$$

or,

$$i\hbar \frac{\partial}{\partial t} \psi\,(q, t) = H\,(t_0)\,\psi\,(q, t). \tag{3·4}$$

4. When the state is not stationary, *i.e.*, it is not a characteristic state of either energy or of an observable, which is a constant of the motion, we may assume the change of state to be continuous so that in a small interval of time δt, the state remains stationary. The representations $\psi\,(q, t)$ and $\psi\,(q, t + \delta t)$ of the state, at t and $t + \delta t$, will therefore, be connected by the infinitesimal unitary transformation

$$U\,(t + \delta t, t) = \exp\,(\,-\,iH\,(t)\,\delta\,t/\hbar) = 1 - (i/\hbar)\,H\,(t)\;\delta t \tag{4·1}$$

upto the first power of $\delta t/\hbar$.

We, therefore, have

$$\psi(q, t + \delta t) = U\,(t + \delta t, t)\,[\psi\,(q, t)] = (1 - iH\,(t)\;\delta t/\hbar)\,[\psi\,(q, t)],$$

whence, on passing to the limit as δt tends to zero, we have

$$\frac{\partial}{\partial t}\,\psi\,(q, t) = -\,(i/\hbar)\,H\,(t)\,\psi\,(q, t). \tag{4·2}$$

5. Thus, both in the case of stationary and of non-stationary (*i.e.* time-dependent) states the law of evolution of the state in time is described by Schrodinger's equation (1·1).

We have derived the equation on the assumption of the complementarity between time and energy and of the unitary nature of the transformation of representations.

REFERENCES

1. Dasvarma, R. L. "On the Characteristic States of Complementary Observables" (communicated to National Academy of Sciences, India).

2. Neumann, J. von. Mathematische Grundlagen der Quantenmechanik, Springer, Berlin (1932).

3. Dasvarma, R. L. "On the Role of Operators as Observables in Quantum Theory, Parts I and II (to be published in the Proceedings of N. A. Sc.)

SOME EXPANSIONS IN BESSEL FUNCTIONS INVOLVING GENERALISED HYPERGEOMETRIC FUNCTIONS

By

H. M. SRIVASTAVA

Department of Mathematics, University of Jodhpur, Jodhpur

[Received on 10th August, 1964]

ABSTRACT

A number of expansions associated with products of Bessel functions, and of Bessel and Gauss's hypergeometric functions, involving certain hypergeometric functions of two variables with higher parameters and of more variables have been given. Some of these expansions are generalisations of the results previously given by Bailey, Fox and Rice.

1. INTRODUCTION

As long ago as 1904 Bateman gave the well-known expansion (see [11], p. 370)

$$(1.1) \qquad \tfrac{1}{2}z\, J_\mu\,(z \cos\phi \cos\Phi).\ J_\nu\,(z \sin\phi \sin\Phi)$$

$$= \cos^\mu\phi \cos^\mu\Phi \sin^\nu\phi \sin^\nu\Phi \sum_{n=0}^{\infty} (-)^n\,(\mu+\nu+2n+1)\, J_{\mu+\nu+2n+1}\,(z)$$

$$\times \frac{\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)}{n!\,\Gamma(\mu+n+1)\,\{\Gamma(\nu+1)\}^2}.\ {}_2F_1\,(-n, \mu+\nu+n+1\,;\, \nu+1;\, \sin^2\phi)$$

$$\times\ {}_2F_1\,(-n, \mu+\nu+n+1\,;\, \nu+1\,;\, \sin^2\Phi),$$

valid for all values of μ and ν with the exception of negative integral values. Some further expansions in Neumann series were given by Fox ([4], §5), one of which is

$$(1.2) \qquad \left\{ J_\nu\,(\tfrac{1}{2}z) \right\}^2 = \left(\frac{2}{\pi z}\right)^{1/2} \sum_{r=0}^{\infty} \left\{ (2\nu+\tfrac{1}{2}+2r)\, \frac{\Gamma(r+\frac{1}{2})\,\Gamma(r+\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(r+\nu+1)} \times \frac{\Gamma(r+2\nu+\frac{1}{2})}{r!\,\Gamma(r+2\nu+1)}\, J_{2\nu+\frac{1}{2}+2r}\,(z) \right\},$$

and it can be readily seen that Fox's results are not included in Bateman's expansion.

In the year 1935, Rice ([7], §III) gave an expansion of a different type, namely

$$(1.3) \quad J_\mu\,(z \cos\phi\, \cos\Phi)\, J_\nu\,(z \sin\phi \sin\Phi)$$

$$= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\ .\ \cos^\mu\phi \cos^\mu\Phi \sin^\nu\phi \sin^\nu\Phi$$

$$\times \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^r}{r!} J_{\mu+\nu+r}(z)\, F_2(\mu+\nu+1, -r, -r; \mu+1, \nu+1; \cos^2\phi, \sin^2\phi),$$

where F_2 is Appell's double hypergeometric function of the second type, and the same year, Bailey ([2], §4) obtained a number of expansions, one of which is

$$(1.4) \quad (\tfrac{1}{2}z)^{1/2} J_\mu(z\cos^2\phi)\, J_\nu(z\sin^2\phi) = \frac{\cos^{2\mu}\phi\, \sin^{2\nu}\phi}{\Gamma(\nu+1)}$$

$$\times \sum_{n=0}^{\infty} \frac{(-)^n(\mu+\nu+2n+\frac{1}{2})(\nu+\frac{1}{2})_n\, \Gamma(\mu+\nu+n+\frac{1}{2})}{n!\, \Gamma(\mu+n+1)}$$

$$\times J_{\mu+\nu+2n+\frac{1}{2}}(z)\, {}_2F_1(-2n, 2\mu+2\nu+2n+1; 2\nu+1; \sin^2\phi),$$

whose particular cases ([2], (4.2) and (4.3)) are generalisations of Fox's results already referred to.

In this paper I give various expansions in products of Bessel functions, and of Bessel and Gauss's hypergeometric functions, involving certain generalised hypergeometric functions.

2. EXPANSIONS INVOLVING DOUBLE HYPERGEOMETRIC FUNCTIONS WITH HIGHER PARAMETERS

Set A

We know that ([6], p. 25)

$$(2.1) \quad (\tfrac{1}{2}z)^{\mu+1} = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+2)} e^z \sum_{m=0}^{\infty} (-)^m \frac{(\mu+m+1)\,\Gamma(2\mu+m+2)}{m!} I_{\mu+m+1}(z),$$

μ not a negative integer.

Using (2.1) we find that

$$(\tfrac{1}{2}z)^{k-\mu-\nu}\, e^{-z}\, J_\mu(az)\, J_\nu(bz)$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-)^{r+s}\, e^{-z}\, (\frac{1}{2}z)^{k+2r+2s}}{r!\, s!\, \Gamma(\mu+r+1)\, \Gamma(\nu+s+1)}\, a^{\mu+2r}\, b^{\nu+2s}$$

$$= \sum_{r,s=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^{m+r+s}\, (\frac{1}{2}z)^{r+s}\, a^{\mu+2r}\, b^{\nu+2s}}{m!\, r!\, s!}$$

$$\times \frac{\Gamma(k+r+s)\,(k+r+s+m)\,\Gamma(2k+2r+2s+m)}{\Gamma(\mu+r+1)\,\Gamma(2k+2r+2s)\,\Gamma(\nu+s+1)} I_{k+r+s+m}(z)$$

$$= \frac{a^{\mu} \, b^{\nu} \, \Gamma(k)}{\Gamma(\mu+1) \, \Gamma(2k) \, \Gamma(\nu+1)} \sum_{n=0}^{\infty} \sum_{r,s=0}^{r+s \leqslant n} (-)^n \frac{(n+k) \, \Gamma(n+2k)}{n!} I_{n+k}(z)$$

$$\times (-)^{r+s} \frac{(-n)_{r+s} \, (2k+n)_{r+s}}{r! \; s! \; (k+\frac{1}{2})_{r+s} \, (\mu+1)_r \, (\nu+1)_s} \left(\frac{1}{8} \, a^2 z \right)^r \left(\frac{1}{8} b^2 z \right)^s ,$$

on setting $m = n - r - s$, and we have the expansion

$$(2.2) \quad (\tfrac{1}{2}z)^{k-\mu-\nu} \; e^{-z} \; J_{\mu}(az) \, J_{\nu}(bz)$$

$$= \frac{a^{\mu} \, b^{\nu} \, \Gamma(k)}{\Gamma(\mu+1) \, \Gamma(2k) \, \Gamma(\nu+1)} \sum_{n=0}^{\infty} (-)^n \frac{(n+k) \, \Gamma(n+2k)}{n!} I_{n+k}(z)$$

$$\times F\left[\begin{matrix} -n, \, 2k+n : \\ k+\frac{1}{2} : \mu+1 ; \; \nu+1 ; \end{matrix} \; -\tfrac{1}{8}a^2 z, \; -\tfrac{1}{8}b^2 z \right],$$

the notation for the double hypergeometric function being that given by Burchnall and Chaundy ([3], pp. 112-3) in preference to the one introduced earlier by Kampé de Fériet ([1], p. 150).

Set B

In this set I give three formulae which are limiting cases of the expansions to be given in the following set. The formulae are

$$(2.3) \quad (\tfrac{1}{2}z)^{k-\mu} \; J_{\mu}(az) \; {}_1F_1(k' ; \nu ; -\tfrac{1}{4} b^2 z^2)$$

$$= \frac{a^{\mu}}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(k+2n) \, \Gamma(k+n)}{n!} J_{k+2n}(z)$$

$$\times F\left[\begin{matrix} -n, \, k+n : - ; \, k' ; \\ \mu+1 \; ; \; \nu ; \end{matrix} \; a^2, \, b^2 \right],$$

$$(2.4) \quad (\tfrac{1}{2}z)^{k-\mu} \; J_{\mu}(az) \; {}_1F_1(k' ; \nu ; -\tfrac{1}{4} b^2 z^2)$$

$$= \frac{a^{\mu} \, \Gamma(k+1)}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} J_{k+n}(z)$$

$$\times F\left[\begin{matrix} -n, \, k+1 : - ; \, k' ; \\ \mu+1 \; ; \; \nu ; \end{matrix} \; a^2, \, b^2 \right]$$

and

$$(2.5) \quad (\tfrac{1}{2}z)^{k-\mu} \; J_{\mu}(az) \; {}_1F_1(\nu - k' ; \nu ; \tfrac{1}{4} b^2 z^2)$$

$$= \frac{a^{\mu}\ \Gamma(k)}{\Gamma(\mu+1)\ \Gamma(2k)} \sum_{n=0}^{\infty} (-)^n \frac{(k+n)\ \Gamma(2k+n)}{n!} I_{k+n}(z)$$

$$\times\ F \left[\begin{matrix} -n,\ 2k+n : - ;\ k' ; \\ k+\tfrac{1}{2} : \mu+1\ ;\ \nu ; \end{matrix}\ -\tfrac{1}{4}a^2 z,\ -\tfrac{1}{4}b^2 z \right],$$

in the notation already referred to.

To prove (2.3), we expand the functions on the left in powers of z and use Neumann expansion ([11], § 5.2)

$$(2.6) \qquad (\tfrac{1}{2}z)^{\nu} = \sum_{m=0}^{\infty} \frac{(\nu+2m)\ \Gamma(\nu+m)}{m!} J_{\nu+2m}(z).$$

Similarly, by using the formula ([11], p. 141 (7))

$$(2.7) \qquad (\tfrac{1}{2}z)^{\nu} = \Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^m}{m!} J_{\nu+m}(z),$$

we obtain (2.4). The formula (2.5) can similarly be proved by using the known expansion (2.1).

Set C

By expanding the first member in powers of z and applying the formulae (2.1) (2.6) and (2.7) in turn, we get the expansions

$$(2.8) \quad (\tfrac{1}{2}z)^{k-\mu} J_{\mu}(az)\ {}_2F_1(\alpha, \beta\ ;\ \nu\ ; -\tfrac{1}{4}b^2 z^2)$$

$$= \frac{a^{\mu}}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(k+2n)\ \Gamma(k+n)}{n!} J_{k+2n}(z)$$

$$\times\ F \left[\begin{matrix} -n,\ k+n : - ;\ \alpha,\ \beta\ ; \\ \mu+1\ ;\ \nu\ ; \end{matrix}\ a^2,\ b^2 \right],$$

$$(2.9) \quad (\tfrac{1}{2}z)^{k-\mu} J_{\mu}(az)\ {}_2F_1(\alpha, \beta\ ;\ \nu ;\ -\tfrac{1}{4}b^2 z^2)$$

$$= \frac{a^{\mu}\ \Gamma(k+1)}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} J_{k+n}(z)$$

$$\times\ F \left[\begin{matrix} -n,\ k+1 : - ;\ \alpha,\ \beta ; \\ \mu+1\ ;\ \nu\ ; \end{matrix}\ a^2,\ b^2 \right],$$

$$(2.10) \quad (\tfrac{1}{2}z)^{k-\mu}\ e^{-z}\ J_{\mu}(az)\ {}_2F_1(\alpha, \beta\ ;\ \nu\ ;\ -\tfrac{1}{4}b^2 z^2)$$

$$= \frac{a^{\mu}\ \Gamma(k)}{\Gamma(\mu+1)\ \Gamma(2k)} \sum_{n=0}^{\infty} (-)^{n} \frac{(k+n)\ \Gamma(2k+n)}{n!} I_{k+n}(z)$$

$$\times F\left[\begin{matrix} -n, 2k+n : -; \alpha, \beta ; \\ k+\frac{1}{2} : \mu+1 ; \nu ; \end{matrix} -\tfrac{1}{8}a^2 z, -\tfrac{1}{8}b^2 z\right],$$

and as already pointed out, the formulae (2.3) to (2.5) are limiting cases of these expansions.

For further generalisations of the results in this section see [9] and [10]. The latter also incorporates a number of interesting expansions in products of Bessel functions which are substantially the consequences of the known formula ([11], p. 151)

$$\left(\tfrac{1}{2}z\right)^{\mu+\nu} = \frac{\Gamma(\mu+1)\ \Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} \sum_{m=0}^{\infty} \frac{(\mu+\nu+2m)\ \Gamma(\mu+\nu+m)}{n!} J_{\mu+m}(z)\ J_{\nu+m}(z)$$

3. EXPANSIONS INVOLVING SARAN'S F_R

Application of the formulae (2.6) and (2.7), again, leads to the expansions

$$(3.1) \qquad (\tfrac{1}{2}z)^{k-\mu-\nu} J_{\mu}(az)\ {}_2F_1(\lambda, k' ; \nu+1 ; b^2)\ J_{\nu}(cz)$$

$$= \frac{a^{\mu}\ c^{\nu}}{\Gamma(\mu+1)\ \Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(k+2n)\ \Gamma(k+n)}{n!} J_{k+2n}(z)$$

$$\times F_R(-n, \lambda, -n, k+n, k', k+n ; \mu+1, [\nu+1], [\nu+1] ; a^2, b^2, c^2)$$

and

$$(3.2) \quad (\tfrac{1}{2}z)^{k-\mu-\nu} J_{\mu}(az)\ {}_2F_1(\lambda, k' ; \nu+1 ; b^2)\ J_{\nu}(cz)$$

$$= \frac{a^{\mu}\ c^{\nu}\ \Gamma(k+1)}{\Gamma(\mu+1)\ \Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} J_{k+n}(z)$$

$$\times F_R(-n, \lambda, -n, k+1, k', k+1; \mu+1, [\nu+1], [\nu+1] ; a^2, b^2, c^2),$$

where F_R is a hypergeometric function of three variables defined by Saran ([8], p. 78 (1.8)).

4. PRODUCT OF THREE BESSEL FUNCTIONS

I now give two expansions which are associated with products of three Bessel functions, namely

$$(4.1) \qquad (\tfrac{1}{2}z)^{k-\lambda-\mu-\nu} J_{\lambda}(az)\ J_{\mu}(bz)\ J_{\nu}(cz)$$

$$= \frac{a^{\lambda} \; b^{\mu} \; c^{\nu}}{\Gamma(\lambda+1) \; \Gamma(\mu+1) \; \Gamma(\nu+1)} \sum_{n=0}^{\infty} (k+2n) \frac{\Gamma(k+n)}{n!} \; J_{k+2n} \; (z)$$

$$\times \; F_C \; (-n, k+n \; ; \lambda+1, \mu+1, \nu+1 \; ; a^2, b^2, c^2)$$

and

$$(4.2) \qquad (\tfrac{1}{2}z)^{k-\lambda-\mu-\nu} \; J_{\lambda} \; (az) \; J_{\mu} \; (bz) \; J_{\nu} \; (cz)$$

$$= \frac{a^{\lambda} \; b^{\mu} \; c^{\nu} \; \Gamma(k+1)}{\Gamma(\lambda+1) \; \Gamma(\mu+1) \; \Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} \; J_{k+n} \, (z)$$

$$\times \; F_C \; (-n, k+1 \; ; \lambda+1, \mu+1, \nu+1 \; ; a^2, b^2, c^2)$$

where F_C is Lauricella's triple hypergeometric function of the third type (see, *e.g.*, [5], p. 111).

To prove (4.1), we expand the Bessel functions in powers of z and use the formula (2.6); the expansion (4.2) follows similarly if we use the formula (2.7).

5. GENERALISATIONS

The results of the preceding section can be generalised, and we have

$$(5.1) \quad (\tfrac{1}{2}z)^{k-\nu_1-\nu_2-\cdots-\nu_n} \; J_{\nu_1} \; (a_1 z) \, J_{\nu_2} \; (a_2 \, z) \; \ldots\ldots \; J_{\nu_n} \, (a_n \, z)$$

$$= \frac{a_1^{\nu_1} \, a_2^{\nu_2} \ldots a_n^{\nu_n}}{\Gamma(\nu_1+1) \; \Gamma(\nu_2+1) \ldots \Gamma(\nu_n+1)} \sum_{r=0}^{\infty} (k+2r) \frac{\Gamma(k+r)}{r!} \quad J_{k+2r} \quad (z)$$

$$\times \; F_C^{(n)} \; (-r, k+r \; ; \nu_1+1, \nu_2+1, \ldots, \nu_n+1 \; ; a_1^2, a_2^2, \ldots, a_n^2)$$

and

$$(5.2) \quad (\tfrac{1}{2}z)^{k-\nu_1-\nu_2-\cdots-\nu_n} \; J_{\nu_1} \, (a_1 z) \; J_{\nu_2} \, (a_2 z) \ldots \ldots J_{\nu_n} \, (a_n z)$$

$$= \frac{a_1^{\nu_1} \, a_2^{\nu_2} \ldots a_n^{\nu_n} \Gamma(k+1)}{\Gamma(\nu_1+1) \; \Gamma(\nu_2+1) \ldots \Gamma(\nu_n+1)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^r}{r!} \; J_{k+r} \, (z)$$

$$\times \; F_C^{(n)} \; (-r, k+1 \; ; \nu_1+1, \nu_2+1, \ldots, \nu_n+1 \; ; \; a_1^2, a_2^2, \ldots, a_n^2),$$

which can readily be proved.

It is easy to see that Bailey's results ([2] §3) are particular cases of the expansions given in this and the preceding sections, and therefore, following Bailey, Fox's results and Rice's formula quoted in § 1 above can be deduced fairly easily from these expansions.

ACKNOWLEDGEMENTS

The author is extremely grateful to Vice-Chancellor (Dr. P. L. Srivastava of Bihar University for his constant encouragement. The interest shown by Dr. S. Saran in the preparation of this paper is also thankfully acknowledged.

REFERENCES

[1] Appell, P. et Kampe de Feriet, J. "Fonctions hypergeometriques et hyperspheriques, Polynomes d' Hermite", Gauthier-Villars, Paris (1926).

[2] Bailey, W. N., *Quart. J. of Math. (Oxford)*, 6, 233 (1935).

[3] Burchnall, J. L. and Chaundy, T. W., *Quart. J. of Math.* (Oxford), 12, 112 (1941).

[4] Fox, C., *Proc. London Math. Soc.*, 26, 201 (1927).

[5] Lauricella, G., *Rendiconti Circ. mat. Palermo*, t. VII, 111 (1893).

[6] Luke, Y. L. "Integrals of Bessel functions", McGraw-Hill, New York (1962).

[7] Rice, S. O., *Quart. J. of Math.* (Oxford), 6, 52 (1935).

[8] Saran, S., *Ganita*, 5, 77 (1954).

[9] Srivastava, H. M., "Some expansions of generalized Whittaker functions" : *Proc. Cambridge Philos. Sac.*. 61, 895 (1965).

[10] Srivastava. H. M., "Some expansions in products of hypergeometric functions" : *Proc. Cambridge Philos. Soc.*, 62, 245 (1966).

[11] Watson, G. N. "Theory of Bessel funciions", *Second edition*, Cambridge and New York (1944).

EFFECT OF ORGANIC SUBSTANCES ON NITRIFICATION BY NITROSOMONAS. PART II

By

S. P. TANDON *and* R. C. RASTOGI

Chemistry Department, University of Allahabad, Allahabad

[Received on 28th September, 1965]

ABSTRACT

The effect of D mannose, D-galactose, L-xylose and L-arabinose on nitrite formation by Nitrosomonas has been studied. The results indicate that all these sugars are non-toxic to the bacteria and stimulate their growth and activity.

In a previous communication, Part I, the results obtained in connection with our study of the effect of D-glucose and D-fructose on the formation of nitrite by Nitrosomonas were given. In this paper the results obtained from the study of the effect of D-mannose, D-galactose, L-xylose and L-arabinose on nitrite formation by Nitrosomonas are given.

The experimental procedure adopted was the same as described in Part I.

TABLE 3A

Nitrosification in the presence of D-mannose

(*i*) Volume of the culture medium taken : 80 ml
(*ii*) Volume of the enriched culture (inoculum) added : 1 ml
(*iii*) Volume of the ammonium sulphate added : 1 ml

No.	Amount of D-mannose added to the medium (in mg)	Amount of D-mannose left at different intervals of time (mg./litre)							
		Time in Hours							
		48	96	144	192	240	288	336	384
1.	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2.	5·00	,,	,,	,,	,,	,,	,,	,,	,,
3.	10·00	6·56	,,	,,	,,	,,	,,	,,	,,
4.	20·00	11·65	3·645	,,	,,	,,	,,	,,	,,
5.	25·00	20·412	13·122	5·103	,,	,,	,,	,,	,,
6.	30·00	23·328	17·50	7·29	,,	,,	,,	,,	,,
7.	40·00	34·99	24·057	10·206	4·374	,,	,,	,,	,,
8.	50·00	41·553	36·45	27·70	19·683	9·477	,,	,,	,,
9.	60·00	54·675	47·38	32·805	27·7	15·30	6·56	,,	,,
10.	80·00	71·44	67·80	56·133	49·57	35·721	23·33	11·664	4·374
11.	100·00	90·40	83·106	72·90	66·339	52·49	39·66	29·16	16·767

Control = Containing no mannose.

TABLE 3B

Nitrosification in the presence of D-mannose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of D-mannose added to the medium (in mg)	Nitrite formed at different intervals of times (mg/litre) Time in Hours						
		48	96	144	192	240	288	336
1.	Control	2·852	7·682	23·00	46·00	105·34	191·68	239·66
2.	5·00	1·104	4·784	16·426	46·00	126·50	255·56	287·50
3.	10·00	0·920	3·496	11·50	35·88	105·34	230·00	324·024
4.	20·00	–	1·380	4·140	18·40	69·00	115·00	150·78
5.	25·00	–	–	3·312	12·075	38·337	82·142	117·875
6.	30·00	–	–	1·38	7·668	25·556	65·996	88·458
7.	40·00	–	–	–	2·300	8·845	28·75	76·68
8.	50·00	–	–	–	2·300	6·599	19·168	53·038
9.	60·00	–	–	–	–	1·380	3·680	14·669
10.	80·00	–	–	–	–	–	1·380	4·600
11.	100·00	–	–	–	–	–	–	–

Control = Containing no mannose.

TABLE 4A

Nitrosification in the presence of D-galactose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of D-galactose added to the medium (in mg)	Amount of D-galactose left at different intervals of time Time in Hours							
		48	96	144	192	240	288	336	384
1.	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2.	5·00	"	"	"	"	"	"	"	"
3.	10·00	5·832	"	"	"	"	"	"	"
4.	20·00	13·122	4·374	"	"	"	"	"	"
5.	25·00	15·309	6·561	"	"	"	"	"	"
6.	30·00	23·328	11·664	7·29	"	"	"	"	"
7.	40·00	35·721	14·580	11·664	3·645	"	"	"	"
8.	50·00	41·553	29·60	23·328	16·038	9·477	"	"	"
9.	60·00	53·946	38·637	30·618	22·599	17·996	11·664	6·561	"
10.	80·00	70·713	62·694	54·675	46·656	42·282	35·72	24·786	17·53
11.	100·00	94·777	86·751	80·190	74·358	6·797	56·861	42·925	28·431

Control = Containing no galactose.

TABLE 4B

Nitrosification in the presence of D-galactose

(*i*) Volume of the culture medium taken 80 ml
(*ii*) Volume of the enriched culture (inoculum) added 1 ml
(*iii*) Volume of the ammonium sulphate added 1 ml

No.	Amount of D galactose added to the medium (in mg)	Nitrite formed at different intervals of time (mg/litre) Time in Hours						
		48	96	144	192	240	288	336
1.	Control	1·150	3·1945	9·0814	36·800	55·69	104·572	205·315
2.	5·00	–	1·5131	5·4761	26·1363	63·894	146·965	240·00
3.	10·00	–	–	1·25	8·280	35·384	115·00	268·180
4.	20·00	–	–	0·9426	2·2115	6·7647	31·645	115·00
5.	25·00	–	–	–	1·012	2·400	17·228	67·055
6.	30·00	–	–	–	0·828	1·370	8·00	26·270
7.	40·00	–	–	–	–	–	1·654	6·389
8.	50·00	–	–	–	–	–	1·012	2·0535
9.	60·00	–	–	–	–	–	–	1·4375
10.	80·00	–	–	–	–	–	–	traces
11.	100·00	–	–	–	–	–	–	–

Control = Containing no galactose.

TABLE 5A

Nitrosification in the presence of L (+) xylolse

(*i*) Volume of the culture medium taken 80 ml
(*ii*) Volume of the enriched culture (inoculum) added 1 ml
(*iii*) Volume of the ammonium sulphale added 1 ml

No.	Amount of L (+) xylose added to the medium (in mg)	Amount of L (+) xylose left at different intervals time (mg/litre) Time in Hours							
		48	96	144	192	240	288	336	384
1.	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2.	5·00	2·40	”	”	”	”	”	”	”
3.	10·00	6·015	2·4	”	”	”	”	”	”
4.	20·00	13·234	9·0225	3·61	”	”	”	”	”
5.	25·30	19·850	14·436	7·218	3·006	”	”	”	”
6.	30·00	25·459	20·450	15·48	8·421	”	”	”	”
7.	40·00	36·70	32·481	24·00	17·445	9·024	2·40	”	”
8.	50·00	47·519	42·707	35·49	27·07	14·44	6·015	”	”
9.	60·00	56·541	53·535	47·52	38·50	27·07	20·451	”	”
10.	80·00	77·594	73·984	66·97	60·150	49·94	37·293	15·048	6·015
11.	100·00	98·044	94·436	86·01	78·195	68·57	57·142	44·51	32·481

Control = Containing no Xylose.

TABLE 5B

Nitrosification in the presence of L (+) xylose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculm) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of L (+) xylose added to the medium (in mg)	Nitrite formed at different intervals of time (mg/litre) Time in Hours 48	96	144	192	240	288	336
1.	Control	0·6440	1·0120	2·1296	5·5168	22·573	67·947	115·30
2.	5·00	0·5780	0·690	1·1372	4·4230	28·750	87·954	194·09
3.	10·00	–	0·5780	0·828	1·2611	5·227	30·50	117·34
4.	20·00	–	–	0·306	1·0455	2·916	11·50	66·503
5.	25·00	–	–	–	0·5060	1·1690	6·389	26·285
6.	30·00	–	–	–	–	0·414	1·315	5·555
7.	40 00	–	–	–	–	–	0·920	1·667
8.	50·00	–	–	–	–	–	–	0·575
9.	60·00	–	–	–	–	–	–	–
10.	80·00	–	–	–	–	–	–	–
11.	100·00	–	–	–	–	–	–	–

Control = Containing no xylose.

TABLE 6A

Nitrosification in the presence of L (+) arabinose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphale added = 1 ml

No.	Amount of L (+) arabinose added to the medium (in mg)	Amount of L (+) arabinose left at different intervals of times (mg/litre) Time in Hours 48	96	144	192	240	288	336	384
1.	Control	Nil	Nil	Nil	Nil	Nil	Nil	Nil	Nil
2.	5·00	,,	,,	,,	,,	,,	,,	,,	,,
3.	10·00	4·09	,,	,,	,,	,,	,,	,,	,,
4.	20·00	17·523	10·51	5·257	,,	,,	,,	,,	,,
5.	25·00	21·027	16·35	9·929	5·257	,,	,,	,,	,,
6.	30·00	25·116	22·78	18·107	12·266	3·50	,,	,,	,,
7.	40·00	34·46	29·205	24·532	16·94	8·178	,,	,,	,,
8.	50·00	46·73	43·23	37·382	30·37	22·78	14·60	4·09	,,
9.	60·00	57·242	53·153	49·649	42·639	35·046	27·452	19·275	9·929
10.	80.00	77·69	72·428	66 587	60·746	52·57	43·23	34·462	24·532
11.	100·00	98·13	94·042	89·96	82·358	75·349	68·34	59·678	46·73

Control = Containing no arabinose.

TABLE 6B

Nitrosification in the presence of L (+) arabinose

(*i*) Volume of the culture medium taken = 80 ml
(*ii*) Volume of the enriched culture (inoculum) added = 1 ml
(*iii*) Volume of the ammonium sulphate added = 1 ml

No.	Amount of L (+) arabinose added to the medium (in mg)	Nitrite formed at different intervals of times (mg/litre) Time in Hours 48	96	144	192	240	288	336
1.	Control	0·8280	2 613	5·210	22·916	39·90	62·50	104·55
2.	5·00	0·6900	1·670	3·571	17·857	41·667	78·125	125·00
3.	10·00	–	0·920	1·150	7·590	28·210	57·50	113·60
4.	20·00	–	–	0·69	1·4375	5·834	10·455	64·09
5.	25·00	–	–	–	0·575	1·150	5·2084	29·00
6.	30 00	–	–	–	–	0·920	2·770	9·375
7.	40·00	–	–	–	–	–	1·15	4·600
8.	50·00	–	–	–	–	–	–	0·920
9.	60·00	–	–	–	–	–	–	–
10.	80·00	–	–	–	–	–	–	–
11.	100·00	–	–	–	–	–	–	–

Control = Containing no arabinose.

Discussion of Results

We had previously observed that D-glucose and D-fructose[1] instead of being toxic to nitritying bacteria serve as better food material and stimulate their growth when presented in proper amount

The results of our study relating to the effect of D-mannose, D-galactose, L-xylose and L-arabinose on nitrite formers (tables 3A, 3B to 6A and 6B) indicate that these sugars also, like glucose and fructose, stimulate the growth and activity of nitrite formers provided they are present in small amounts. Thus these results further confirm that organic material in general is not toxic to nitritying bacteria as has been believed so for.

REFERENCE

1. Tandon, S. P. and Rastogi, R. C. *Proc. Nat. Acad. Sci. India*, 35 : 29 36, (1965).

ON THE ROD SHAPE NATURE OF ZIRCONIUM GLUTARATE COLLOIDAL PARTICLES

By

A. MUKHERJI, R. K. SHINGHAL *and* S. P. MUSHRAN

Chemical Laboratories, University of Allahabad, Allahabad

[Received on 2nd February, 1966]

ABSTRACT

Employing stability, rheological and light scattering data the nature of the gel forming zirconium glutarate colloidal particles has been investigated. The non-spherical nature of the colloidal particles has been established and it has been further shown that the zirconium glutarate particles are rod shaped.

INTRODUCTION

In a series of publications Mushran and co-workers[1] have described the preparation and properties of several metal organo colloidal dispersions. In a recent publication Mukherji and Mushran[2] have described the preparation of zirconium glutarate gels obtained from sols of different purity and particle size. It is well known that the gelation process is controlled by several factors and the shape and size of colloidal micelles play a very important role. In this communication we are presenting our investigations on the stability, rheological properties and light scattering studies of gel forming zirconium glutarate sols with a view to exmamine the shape of colloidal particles undergoing gelation.

EXPERIMENTAL

Three samples of zirconium glutarate sols were obtained by mixing a fixed quantity of solutions of zirconium nitrate and different amounts of sodium glutarate keeping the total volume constant equal to 300·00 mls. in each case. The sols were purified by dialysis at room temperature and preserved in Jena bottles at low temperature. The zirconium content in each of the sols were estimated by the usual methods and the amount of the metal ion was kept constant by adding distilled water wherever necessary. Sols A, B and C had the following composition :

Volume of 0·4 M zirconium nitrate = 150 mls.

Volume of 0·1 M sodium glutarate

in sol (A) = 95·00 ml.
in sol (B) = 97·50 ml.
in sol (C) = 100·00 ml.

The sols were dialysed for 70 hours.

Stability Measurements.—For the determination of the stability of the sols, the gelling time employing potassium chloride as gelating electrolyte, was found by the method described by Bose and Mushran.[3] The stability factor log W was obtained from the relation log W = log $t/\bar{t}$ where 't' and '$\bar{t}$' represent the gelling time for slow and rapid gelation.

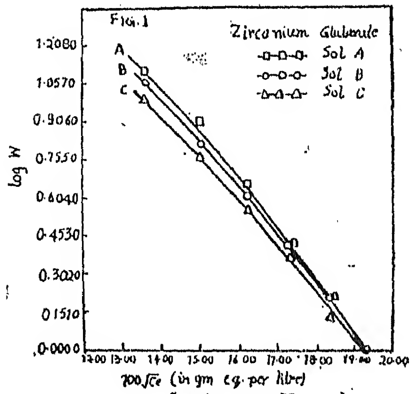
According to Packter[4] the coagulation of several dilute sols with coagulant electrolytes takes place by a process which involves the perpendicular orientation of adjacent particles. For sols with rod like particles the process of coagulation by monovalent ions is governed by the following relation :

$$\frac{d \log W}{d\sqrt{Ce}} = 300\ \gamma^2$$

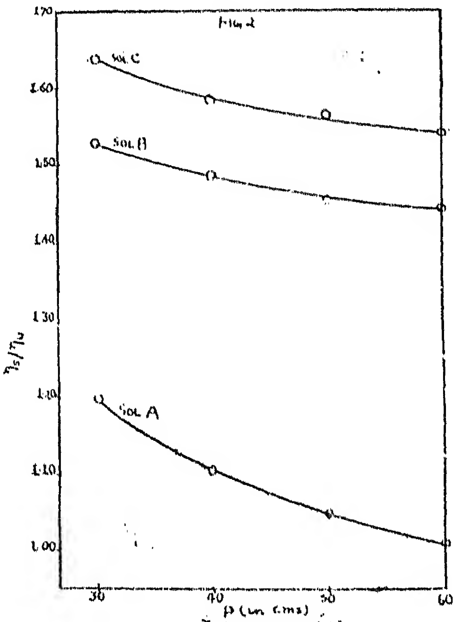
where log W represents the stability factor, Ce is the concentration of the coagulant electrolyte and γ depends upon the value of surface potential ψ. Working with several sols Packter has established further the linear relation between log W and Ce given by the expression :

$$\log W = C_1 \gamma^4 - C_2 \gamma^2 \sqrt{Ce}$$

In Fig. 1 is represented our stability data with the sol samples A, B and C. It is seen that the plot of log W and $\sqrt{Ce}$ is a straight line suggesting the applicability of Packter's equation. Since the Packter equation involves rod shaped particles it is evident that the zirconium glutarate particles, in all probability have a rod shaped structure. The aggregation and gelation of the colloidal particles thus takes place by a process of charge neutralisation followed by a process of perpendicular orientation of the rod shaped aggregated particles. The solvent is imbibed between the adjacent rod shaped particles giving rise to gels of sufficient mechanical strength.



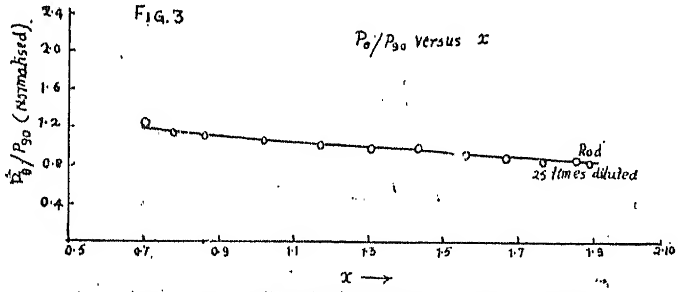
log W versus √Ce



η_s/η_u versus 'p'

Rheological Studies.—Rheological measurements with sols were made by the help of an Ostwald viscometer with suitable modifications,[5] so that the value of specific viscosity at different shearing stresses could be accurately obtained. In Fig. 2 are plotted the values of specific viscosity as a function of pressure

(driving force) for the three different samples of zirconium glutarate sols. The investigations were carried out at a temperature of 30 ± 0·1°C. It will be seen that the plots are not straight lines but are curvilinear in nature and the specific viscosity decreases to a considerable extent with an increase in the value of the driving force. It is therefore evident that the particles are undoubtedly not spherical in shape, due to which Einstein's viscosity relation for spherical particles, *viz.*, $\eta_s = \eta_w (1 + K \phi)$ is not obeyed, but have a different shape. The fact that the viscosity decreases with an increase in the driving force, suggest an orientation of the particles, with a definite rod like nature, towards the direction of the flow. In normal condition the rod shaped particles of zirconium glutarate sol because of Brownian movement are scattered throughout the solution in complete disorder. Under the influence of a sufficiently great shear stress, the rod shaped particles are oriented along the stream lines and this orientation causes the decrease in the specific viscosity of the colloidal dispersion. Rheological data therefore indicates the non-Newtonian characteristic of the gel forming ziroconium glutarate sol and clearly indicates that the particle shape is rod like.



Lines represent theoretical curves and circles represents experimental points

Light Scattering Measurements.—For light scattering investigations the gels of zirconium glutarate were obtained by metathetical interaction of zirconium nitrate and sodium glutarate solution of suitable concentration. A Bryce-Phoenix universal light scattering photometer was employed for all scattering measurements and light of wavelength 446 m μ was used since the system, *i.e.*, zirconium glutarate and water absorbs very little light between 540 – 600 mμ. All zirconium nitrate and sodium glutarate solution of A. R. quality were filtered several times to make them free from dust particles. The intensity of the scattered light was measured at different angles in the cylinderical cell having one side frosted. The intensities for these angles were subjected to volume correction for the cell by multiplying by sin θ and were normalised to account for the variation due to the resolution of polarised component by dividing by $1 + \cos^2 \theta$. The ratio of the intensities of the light scattered at a given angle to that scattered at 90° was determined for the angle used, the observed ratio agreed satisfactorily with that calculated from the theoritical curve given by Debye[6] for rod shaped particles as follows :

$$P (\theta) = 1/x \int_0^{2x} \frac{\text{Sin } W}{W} \, d w - \left(\frac{\text{Sin } x}{x} \right)^2$$

where $x = \dfrac{KSL}{2}$, $K = 2 \pi/\lambda'$, where λ' is the wave-length of light in solution and $S = 2 \text{ Sin } \theta/2$ and 'W' is variable over which the integration is made.

Fig. 3 represents the plot of $P(\theta)/P(90)$ employing a mixture of 2·0 mls (0·20 M zirconium nitrate) and 1·0 ml. (0·20 M sodium glutarate) in a total volume of 10·0 mls. The mixture was diluted to five times and allowed to stand for 45 minutes after which the observations were recorded. It will be seen from Fig. 3 that for the angles used, the observed ratio agreed satisfactorily with that calculated from the theoritical curve (variation of $P(\theta)$ with x). From the above consideration it can therefore be clearly concluded that the particles of the gel forming glutarate sol which is produced in a transitory stage as a result of interaction between zirconium nitrate and sodium glutarate solution are rod shaped in nature.

From measurements of turbidity, dysymmetry values during the process of gelation, the length of the colloidal particles have also been calculated and such results will be communicated in a subsequent publication.

ACKNOWLEDGMENT

This work has been supported by a Government of India Scholarship to A. M. and by Junior Reseach Fellowship of C. S. I. R. to R. K. S.

REFERENCES

1. Mushran, S. P. and co-workers. *Nature*, 163, 879 (1949) ; *Current Science*, 26, 47 (1957) ; *Naturwiss*, 45, 514 (1958) ; *Kolloid-Z*, 171, 31 (1960).
2. Mukherji, A. and Mushran, S. P. *J. Ind. Chem. Soc.*, 40, 245 (1963).
3. Bose, S. K. and Mushran, S P. *Proc. Natl. Acad. Sci. India*, 27A, 21 (1958).
4. Packter, A. *Kolloid-Z*, 154, 62 (1957).
5. Bose, S. K. and Mushran, S. P. *Rheologica Acta*, 3, 18 (1963).
6. Debye, P. *J. Phy. Chem.* 51, 18 (1947).

PHYSICO-CHEMICAL STUDIES OF PALLADIUM (II) CITRATE COMPLEX

By

J. N. MATHUR, C. B. GANDHI *and* S. N. BANERJI

Chemistry Department University of Saugar, Sagar (M. P.)

[Received on 22nd November, 1965]

ABSTRACT

Stoichiometry and stability of Palladium Citrate complex has been studied here. The composition of the complex has been determined by Job's continuous variation[1] and mole ratio[2] methods and the stability constant of the complex has been calculated by Job's method using non-equimolar solutions and also by mole ratio[3] method. The index property selected for study was absorption of the solutions. A 1 : 1 (metal : ligand) complex is obtained which is stable between the pH range of 4–5. The apparant stability constant is $32 \cdot 5 \times 10^6$ as determined by the above methods.

Extraction of Citrate and tartrate complexes of metals in the presence of diisoamylamine has been reported by I. V. Pyatnitskii and R. S. Kharchenko[4] but no light has been thrown on the stability of the complexes. It was therefore, thought worthwhile to make a complete study of the composition and stability of the citrate complex of Palladium (II).

EXPERIMENTAL

Palladium chloride (Johnson, Mathey & Co.) solution was prepared by dissolving in water and standardising it by precipitating with dimethylglyoxime.[5] Citric acid (B. D. H. Analar) was taken and its trisodium salt was prepared by the addition of equivalent amount of sodium carbonate (G. R., S. Merck) and boiling the solution to expel all carbon dioxide.

All the mixtures were kept in a thermostat for about 5 hours, to attain the equilibirium. All the measurements were carried out at 30 $\pm$ 0·5°C.

Absorption measurements were made on a Beckman D. U. quartz Spectrophotometer. This instrument incorporates a hydrogen tube source (for a range 220–350m μ) or tungsten lamp source (for wave length more than 350m μ), a quartz littro type monochromator with 30° prism, a photocell and an amplifier. Since all the solutions were almost colourless, hydrogen lamp was used.

Four matched silica cells of 1·0 cm. light path were used for the specimen and the solvent. They were cleaned thoroughly and dried every time before introducing a new sample into them. The optical density measurements were carried out as usual.

For actual experiments wave length selection was first made and suitable wavelengths, where appreciable divergence from the additive value was observed, were selected. The Citrate complex was investigated at 228, 232 and 236m μ.

Job's method of continuous variation has been used taking equimolar solutions of 1×10^{-4} M concentration. The tolal valume of all the mixtures were kept at 10ml. Optical density measurements were made and a graph was plotted

between the difference in optical density and percentage of the ligand added. The maxima (Fig. 2) was obtained at a point indicating 50% of the ligand which clearly shows the compositon of the complex as 1 : 1 (M : L.).

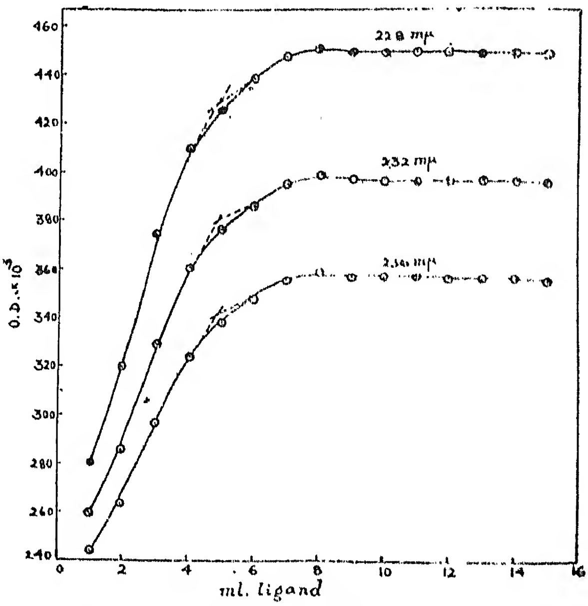


Fig. 1: Composition of the complex

In the mole Ratio method equimolar solutions of palladium chloride and sodium citrate, each of $2{\cdot}5 \times 10^{-4}$M were taken. The total volume of each mixture was kept at 20 ml. 5 ml. of the metal solution was taken in each of the tubes and the ligand solution was added with an increase of 1 ml. for each tube. The total volume of the mixture was made up to 20ml. in each case by the addition of distilled water. The optical density was measured for each of the solutions and a graph was plotted between optical density and the volume (in ml.) of ligand added. A clear break (Fig. 1) was observed at 5 ml. of ligand confirming to a 1 : 1 (M : L) complex.

Evaluation of Stability constant

The stability constant of the complex was calculated by using :

(*i*) Job's method of continuous variation taking non-equimolar solutions. The experimental details remained the same as in the case of determination of the composition. Maxima obtained from these observations (Fig. 3) provide the value of x.

The stability constant was calculated by the following formula :

$$Ks = \frac{(p-1)(1-2x)}{c\,[\,(p+1)\,x-1\,]^2}$$

Here p is given by ratio of concentration of ligand to the concentration of metal. We have taken $p = 2$.

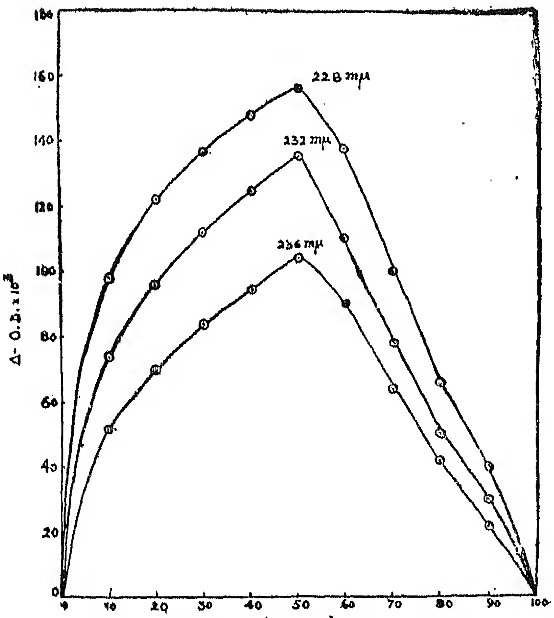


Fig. 2 : Composition of the complex

(*ii*) Mole Rotio method has also been used for the determination of the stability constant.

The dissociation of the complex can be written as :

$$Pd\,R \rightleftharpoons Pd + R$$

c			intitial concentration
$c\,(1-\alpha)$	αc	αc	equilibrium concentration.

where c is the total concentration of the complex in moles per litre assuming no dissociation and α is the degree of dissociation. The equilibirum constant K is given by the equation :

$$K = \frac{\alpha\, c \times \alpha\, c}{c\,(1-\alpha)} = \frac{\alpha^2\, c^2}{c\,(1-\alpha)} = \frac{\alpha^2\, c}{1-\alpha}$$

wherefrom the stability constant will be given by

$$Ks = \frac{1-\alpha}{\alpha^2\, c}$$

where the value of α may be obtained from the upper curve (Fig. 1) by the following relationship.

$$\alpha = \frac{Em - Es}{Em}$$

where *Em* is the maximum absorption from the horizontal portion of the curve, when all the palladium is present in the form of the complex and *Es* is the observed absorption of the stoichiometric molar ratio of the ligand to palladium in the complex.

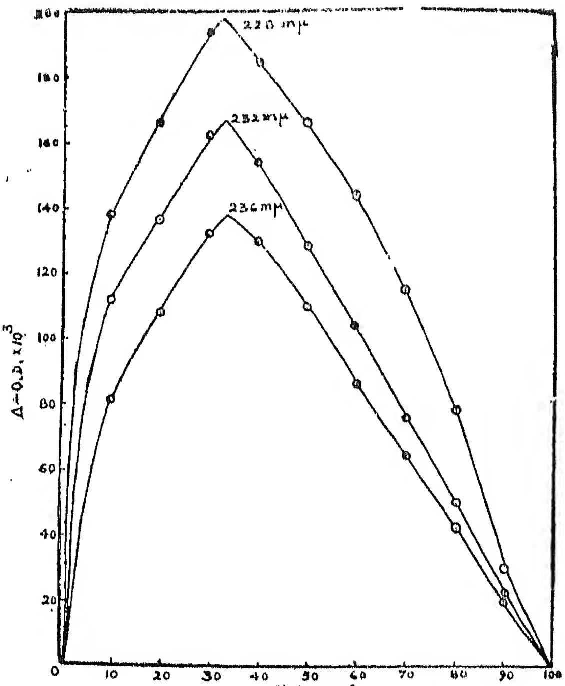


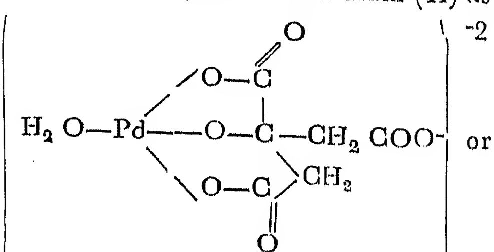
Fig. 3 : Dissociation constant of the complex

The stability constant values obtained by the above methods are 34×10^6 and 31×10^6 respectively. The mean value comes to be $32{\cdot}5 \times 10^6$.

CONCLUSION

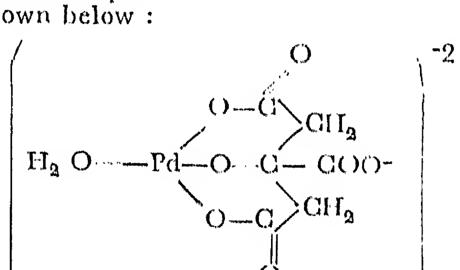
Figures 1 and 2 show that the stoichiometry of the Citrate complex of Palladium is 1 : 1. The value of the stability constant by the two different methods (Fig. 1 and 3) was found to be 34×10^6.

The coordination number of Palladium (II) in soluble complexes is taken to be 4 and the citrate ligand is possibly tridentate. In the complex formation between Palladium (II) and the citrate ligand, three of the four coordination positions of Palladium (II) are occupied by the citrate ligand and the fourth one by a molecule of water as has been reported in the cases of cirtate complexes of thallium[6], lead[7], and antimony[8]. There are two possibilities for the structure of the citrate complex of Palladium (II) as shown below :



'A'

or



'B'

Of the two structures, B seems to be more probable as it contains only six membered rings including Palladium (II), and six membered rings are more stable than the five membered ones. On the other hand, structure A shows one six membered and one five membered ring and should be less stable than that shown by structure B.

REFERENCES

1. Job. *Compt. rend.* 184 : 204, 1066 (1927) ; *Ann. Chem.*, x, 9, 113, (1928).
2. Yoe and Jones. *Ind. Eng. Chem. Anal. Ed.* 16 : 111. (1944).
3. Harvey and Manning *J. Amer. Chem Soc.* 72 : 4488. (1950).
4. Pyatnitskii, I. V. and Kharchenko R. S. *Ukr. Khim. zh.* 28 (9) : 1115 (in Russian) (1962).
5. Vogel. "Text Book of Quantitative Inorganic Analysis", p 445, (1951).
6. Sahu and Bhattacharya. *J. Ind Chem. Soc.* 42 : 247, (1965).
7. Patnaik and Pani. *Ibid.*, 34 : 19, (1957).
8. Das and Pani. *Ibid.*, 32 : 537, (1955).

ON A DERIVATION OF THE PROBABILITY DENSITIES OF POSITION AND MOMENTUM FOR THE HARMONIC OSCILLATOR IN THE GROUND STATE

By

R. L. DASVARMA

Department of Mathematics, L. S. College, Muzaffarpur

[Received on 14th December, 1965]

ABSTRACT

In this paper, the Gaussian nature of the probability distributions for the position and the momentum of a one-dimensional harmonic oscillator, in its ground state, has been deduced from an entropy function. The derivation is based on the classical mechanical functional dependence of the position and the momentum on time, and on the quantum mechanical expression of the energy of the oscillator, making use of Heisenberg's uncertainty relation.

1. The notion of entropy as a measure of uncertainty in the theory of random distributions is defined as follows :

Let X be a continuous random variable, taking on values in the interval $[a, b]$ of the real line. Let $f(x)$ be the density function of the distribution of X, *i.e.*,

$$\int_a^x f(x)\, dx = Pr\,[\,a \leq X \leq x\,].$$

Then the entropy is defined as the functional

$$H(X) = \int_a^b -f(x) \log f(x)\, dx. \qquad \ldots. (1)$$

By maximizing $H(X)$ under given subsidiary conditions, one can obtain the density function $f(x)$. For instance, in the case of the continuous random variable X, defined on the real line $[-\infty, \infty]$, with given second moment b, and zero mean, we have the following conditions for $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1, \qquad \ldots (2)$$

$$\int_{-\infty}^{\infty} x f(x)\, dx = 0, \qquad \ldots. (3)$$

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx = b. \qquad \ldots. (4)$$

By the method of undetermined multipliers,[1] it can be shown that the entropy

$H(X) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx$, is a maximum, if

$$f(x) = (1/2\pi b)^{\frac{1}{2}} \exp(-x^2/2b), \qquad \ldots. (5)$$

i. e., if the distribution is normal Gaussian.

The result can be used to determine the densities of distribution for the coordinate and momentum of the one-dimensional harmonic oscillator in the ground state.

2. Classically, the motion of the one-dimensional oscillator is characterized by the Hamiltonian function

$$H(q,p) = \frac{p^2}{2m} + \frac{m\,\omega^2 q^2}{2}, \qquad \dots . (6)$$

where m is the mass of the oscillator, p, its momentum, q, the coordinate, measured as the distance of the oscillator from the centre of force and $\nu = \frac{\omega}{2\pi}$ is the frequency. For simplicity, let us assume that the oscillator is of unit mass.

The coordinate and the momentum, expressed as functions of time, are

$$q = a \sin \omega t, \; p = a\omega \cos \omega t. \qquad \dots . (7)$$

Averaging over a period of oscillation, $T = 2\pi/\omega$, we have

$$\bar{q} = 0, \bar{p} = 0, \qquad \dots . (8)$$

where $\bar{q}$ and $\bar{p}$ are, respectively, the mean of the coordinate and the momentum. The standard deviation $\sigma(q)$ and $\sigma(p)$, defined by

$$\sigma(q) = \sqrt{\overline{(q-\bar{q})^2}}, \qquad \sigma(p) = \sqrt{\overline{(p-\bar{p})^2}},$$

are, in this case, equal to the square root of the second moments and we have

$$\sigma^2(q) = \overline{q^2}, \; \sigma^2(p) = \overline{p^2}. \qquad \dots . (9)$$

The average of the Hamiltonian function, therefore, over a period oscillation, is

$$\bar{H} = \tfrac{1}{2}\overline{p^2} + \tfrac{1}{2}\omega^2\overline{q^2} = \tfrac{1}{2}\sigma^2(p) + \tfrac{1}{2}\omega^2\sigma^2(q).$$

But the Hamiltonian represents the total energy of the oscillator and is, therefore, constant. Denoting the energy by E, using (9), we have

$$E = \tfrac{1}{2}\sigma^2(p) + \tfrac{1}{2}\omega^2\sigma^2(q). \qquad \dots . (10)$$

3. Now, the energy of the harmonic oscillator in its ground state is

$$E = \tfrac{1}{2}\hbar\omega. \qquad \dots . (11)$$

Therefore, substituting the value of E from (11) into (10), we have

$$\sigma^2(p) + \omega^2\sigma^2(q) = \hbar\omega. \qquad \dots . (12)$$

Further, Heisenberg's uncertainty relation, in this case, is

$$\Delta q \cdot \Delta p = \hbar/2, \qquad \dots . (13)$$

where the uncertainties are defined as the root mean square deviations from the mean values, *i.e.*, the standard deviations $\sigma(q)$ and $\sigma(p)$. Hence, from (13), we have

$$\sigma(q)\,\sigma(p) = \hbar/2. \qquad \dots . (14)$$

From (12) and (14), we have

$$\sigma^2(q) = \hbar/2\omega, \; \sigma^2(p) = \omega\hbar/2. \qquad \dots . (15)$$

4. Because of (9), the second moments of q and p are

$$\overline{q^2} = \hbar/2\omega, \; \overline{p^2} = \omega\hbar/2. \qquad \dots . (16)$$

Hence, both the coordinate and the momentum, considered as continuous random variables, with zero mean and second moments given by (16), must have each normal Gaussian distribution, given by the respective density functions

$$f(q) = \sqrt{\frac{\omega}{\pi \hbar}} \exp(-\omega q^2/\hbar). \qquad \ldots. (17)$$

$$g(p) = \frac{1}{\sqrt{\pi \omega \hbar}} \exp(-p^2/\omega \hbar), \qquad \ldots. (18)$$

for maximum uncertainty in each variable subject to the conditions (12) and (14). The normal distributions (17) and (18) can each be transformed into unit normal distribution by the following substitutions :

$$q = \sqrt{\hbar/2\omega}\ \xi, \quad p = \sqrt{\omega\hbar/2}\ \eta,$$

so that

$$\sigma^2(\xi) + \sigma^2(\eta) = 2, \sigma(\xi)\,\sigma(\eta) = 1, \qquad \ldots. (19)$$

giving

$$\sigma(\xi) = \sigma(\eta) = 1. \qquad \ldots. (20)$$

The density functions for the distributions of ξ and η are

$$\frac{1}{\sqrt{2\pi}} \exp(-\xi^2/2) \text{ and } \frac{1}{\sqrt{2\pi}} \exp(-\eta^2/2).$$

It is readily seen that each is the Fourier transform of the other by the kernel $(\frac{1}{2}\pi)^{\frac{1}{2}} \exp(i\,\xi\,\eta)$ and, therfore, is the characteristic function of the other.

The above derivation has been possible for the ground state of the harmonic oscillator because only in this case the uncertainty relation is an equality relation instead of an inequality relation.

REFERENCE

1. Reza, F. M. An Introduction to Information Theory. *McGraw-Hill* (1961).

CHEMICAL EXAMINATION OF *SESBANIA GRANDIFLORA* (LINN.) PERS. BARK.

Isolation and study of β-sitosterol and β-sitosterol esters of stearic and oleic acids

By

R. D. TIWARI *and* R. K. BAJPAI*

Department of Chemistry, University of Allahabad, Allahabad

[Received on 13th November, 1965]

ABSTRACT

The hot ethanolic extract of the bark of *Sesbania grandiflora* when allowed to stand overnight, deposited a residue which was removed by filtration, the ethanolic extract after concentration was added to a large volume of water. The light brown colored mass so deposited was extracted with petroleum ether, this extract has been found to contain free β-sitosterol and also β-sitosterol esters of stearic and oleic acids.

The isolation and chemical examination of an aliphatic alcohol[1] and a saponin[2] from the leaves of *Sesbania grandiflora* have been described in earlier communications. The trunk bark of *Sesbania grandiflora* when extracted exhaustively with 95% ethanol and the extract allowed to stand overnight at room temperature, deposited a light brown colored product which was separated and the filtrate was concentrated under reduced pressure to a syrupy mass which was added to a large volume of water. As a result of this, a light brown colored mass got deposited which after washing several times with hot water was dried and subsequently extracted with petroleum ether (40–60°) in a Soxhlet extractor.

The petroleum ether extract was concentrated to half its volume and when allowed to stand overnight, it deposited a light yellow residue which was filtered and chromatographed in benzene over Brockmann's alumina. After elution with benzene–acetone (95 : 5), the residue on crystallization from ethanol gave colorless plates m. p. 135–136° [Found : C, 84·20 ; H, 11·89 ; $\alpha_D^{30} = -37°$ (c, 1·5 in chloroform). Calculated for $C_{29}H_{50}O$: C, 84·05 ; H, 12·07%]. The peaks at 3497 (OH), 2899 ($-CH_3$, $-CH_2-$), 1639 cm^{-1} (C = C in steroids) were obtained in I. R. spectrum of the compound in chloroform. Acetate m. p. 126–127°, benzoate m. p. 143–144°, digitonide m. p. 221° (*d*). No depression in melting point of the compound was observed when mixed with an authentic sample of β-sitosterol.

Petroleum ether soluble part :

After separation of β-sitosterol the solvent was completely distilled off at reduced pressure from the petroleum ether extract, when a waxy mass m. p. 62–65° was obtained. It was hydrolysed by refluxing with 0·5N ethanolic potassium hydroxide. After distilling off alcohol the product was poured into cold water, when a dirty white solid separated out. It was chromatographed

---

Present address : Department of Chemistry, College of Science, Ravishankar University, Raipur, M. P.

over alumina followed by crystallization from methanol in the form of colorless plates m. p. 135–136°. The compound was identified as β-sitosterol.

The aqueous alkaline solution was acidified with mineral acids and the fatty acids thus liberated were separated into solid and liquid acids by lead salt alcohol process[3,4]. The acids obtained by the decomposition of solid lead salts were converted into methyl esters which on fractionation under reduced pressure gave two fractions, both of which on saponification were found to contain only stearic acid. The liquid lead salt on decomposition gave a liquid acid which was found to consist of only oleic acid by its equivalent weight, iodine value, and by elaidin test. Thus, the waxy substance, falls under type *B* of group I in the classification of derivatives of higher fatty acids[4] which include ester waxes or more especially sterol or higher alcohol esters of higher fatty acids. This is in agreement with the observation that the bark fats usually consist of oleic acid as the major component along with palmitic or stearic acids (which together constitute not more than one-third of the total acid).[4]

REFERENCES

1. Tiwari, R. D., Bajpai, R. K., and Khanna, S. S., *Arch. Pharm.* 297 : 310, (1964).
2. Tiwari, R. D., and Bajpai, R. K., *Proc. Nat. Acad. Sci. India*, 34 : 239, (1964).
3. Twitchell, E., *J. Ind. Eng. Chem.* 13 : 806, (1921).
4. Hilditch, T. P., 'Chemical Constitution of Natural Fats', pp. 468, 2 : 141, (1949).

DETERMINATION OF IODIDES BY OXIDATION WITH PERMANGANATE

By

SAMEER BOSE

University Chemistry Department, Jabalpur, M. P.

[Received on 20th September, 1965]

ABSTRACT

An iodometric method is described for quick determination of iodide solutions containing 0·5-250 mgms of iodine, by oxidizing with acidified permanganate in presence of 5 gm of ammonium sulphate. The slight excess of oxidant as judged from the pink colour is destroyed by glucose solution and the free iodine titrated with thiosulphate solution to starch end point. Chlorides do not interfere. The method has a precision of 0·5% or more. Metals, like lead, mercury and silver can also be evaluated. For the latter it is recommended in lieu of Volhard method.

In acid solution iodide is oxidized to free iodine by permanganate. Vogel[1] has described potentiometric titration using a combination of platinum and calomel electrodes. The titration to visual end-point has been so far found to be impossible[2] due to the iodine colour. Earlier procedures involving extraction of iodine by shaking with benzene, carbon tetrachloride or other solvents have not much to commend them. Gorbatschaff and Kassatkina[3] saturated the iodide solution with sodium chloride, acidified, and titrated with permanganate. Chloroform served as indicator, being coloured red violet by free iodine until the latter was converted to iodine chloride at the end point. The present worker has evolved a simpler method for carrying this oxidation by visual titration to permanganate colour. The solubility of iodine in water was suppressed by adding large quantities of salts like, ammonium sulphate, magnesium sulphate and aluminium sulphate. Due to considerable reduction in the solubility of iodine and precipitation of the free iodine, the colour, of the reaction mixture is only faint yellow against which the pink colour of permanganate is distinctly perceptible. Apart from the end point this oxidation involves another difficulty and that lies in the conversion of iodine to iodic acid by acidified permanganate, which although is a slow reaction, but can not be ignored. This error can be eliminated to some extent by slow addition of permanganate and quick observation of the pink colour at the end-point. A modified proceedure has also been developed which involves the quick destruction of the excess oxidant by glucose solution and titration of the free iodine after dissolving it in potassium iodide with thiosulphate soltution. Metals which produce insoluble iodides like mercury, silver and lead have also been evaluated.

TABLE 1

Solubility of iodine at 25°C in salt solutions and coagulation of precipitated iodine

Solvent	Mgm of iodine per litre	Coagulation of precipitated iodine
Water	339	–
40% Magnesium sulphate	76	slow
40% Ammonium sulphate	55	quick
40% Aluminium sulphate	51	quick (best)
Saturated Sodium sulphate	25	slow

Reagents :

(1) Standard potassium iodide solution (Merck-extra pure salt used) ; 16·6 gm. of the salt dried at 200°C, dissolved in 1 litre water and stored in the dark.

(2) Stock solution of 0·1N sodium thiosulphate, standardized against potassium dichromate Analar (B. D. H.).

(3) Stock solution of 0·1N potassium permanganate, standardized against thiosulphate solution, stored inside cupboards to prevent action of light.

(4) Salphuric acid solution about 8N, dilute 25 ml of concentrated acid to 100 ml with water.

(5) Glucose solution, prepared fresh by dissolving about 12 gm in 250 ml. water.

(6) Starch solution, 1% amylum starch in water.

(7) Silver nitrate (0·1m), 16·99 gm of Analar (B. D. H.) salt dissolved in 1 litre water.

(8) Lead nitrate (0·05m), 16·6 gm. pure salt dissolved in 1 litre water and acidified with freshly boiled dilute nitric acid.

(9) Mercuric chloride (0·1m), 27·16 gm. pure salt dissolved in 1 litre water.

(10) Ammonium, aluminium, magnesium and sodium sulphates free from iodide and bromide salts were used in powder form.

PROCEDURE

(A) Visual titration to permanganate colour :

Five to 10 ml. of iodide solution containing 25–125 mgm of iodine was pipetted into a 250 ml. conical flask and treated with about 5 gm of solid ammonium-sulphate and 2 ml. of 8N, sulphuric acid solution. The contents were dissolved by shaking and titrated slowly with 0·1N permanganate solution, adding at the rate of 1 ml. per minute with swirling of the flask. The colour changes in the flask were observed by holding it about 8 cm above a white paper kept below the burette and looking down through the sides of the flask but not through the neck. In the beginning a black precipitate of iodine appears which undergoes a marked coagulation just near the end-point. This coagulation was taken as the indication of the approaching end-point. The permanganate was added dropwise with swirling of the flask after the coagulation of iodine precipitate. The final colour change was from faint yellow to practically colour less and then to faint pink. This colour change was observed quickly, as the pink colour produced by one drop excess of permanganate disappeared in half a minutes time. On adding two drops in excess of permanganate the colour stayed longer, but this introduced a slight error. The results obtained by two different workers agreed and are shown in Table II.

1 ml of 0·1N $KMnO_4$ = 12·7 mgm of iodine.

Sodium, magnesium and aluminium sulphates were also used. The latter has better coagulation properties, but takes more time to dissolve. Ammonium salt was preferred being easily available in bromide and iodide freestate. Instead of weighing out 5 gm of the salt every time, a quicker method of measuring it out from a dry test tube with a 5 ml. mark, was adopted.

TABLE II

Temperature 25°C

Determination of potassium iodide by direct titration with $KMnO_4$

(5 ml. to 10 ml of iodide + 5 gm salt + 2 ml. H_2SO_4 8N)

Salt added	KI present	KI found	% error
Ammonium sulphate	33·2 Mgm.	33·4 Mgm.	+0·6
,, ,,	83·0 ,,	83·8 ,,	+1·0
,, ,,	166·0 ,,	166·9 ,,	+0·6
Aluminium ,,	83·0 ,,	83·2 ,,	+0·3
,, ,,	166·0 ,,	166·5 ,,	+0·3
Magnesium ,,	83·0 ,,	83·5 ,,	+0·7
,, ,,	166·0 ,,	167·6 ,,	+1·0

(*B*) *Modified iodometric procedure* :

The excess permanganate can be quickly destroyed by glucose solution and the oxidation of iodine prevented. The glucose solution does not react with free iodine which may be titrated with thiosulphate solution. Five to 10 ml. of iodide solution containing 12·7–250 mgm of iodine was pipetted into a 250 ml stoppered conical flask and treated with 5 gm ammonium sulphate, 2 ml. 8N sulphuric acid and titrated slowly with 0·1N permanganate solution as described earlier, to a slight pink colour. While titrating 50 ml of 5% glucose solution was kept ready in a 100 ml beaker and on seeing a faint pink tinge in the titration flask, it was quickly added with swirling of the flask. The latter was stoppered and kept aside for 2 minutes. About 1 gm of solid potassium iodide was added to the flask and the dissoved iodine titrated with 0·1N, thiosulpate solution to starch end-point. The iodine present in the assay solution of iodide was calculated from the thiosulphate reading. The permanganate reading should exceed the thiosulphate reading by 0·1 – 0·2 ml when both the solutions are of the same starength. This difference was due to the slight excess of permanganate added for producing the pink colour The smaller the excess added, the better were the results. Good results were obtained when glucose solution was added within 15 seconds of the appearance of pink colour, as this completely prevented the oxidation of iodine by excess permanganate added. A duplicate was carried out using lesser amount of the oxidant and the difference between the volumes of permanganate and thiosulphate was only of 0·1 ml. The results are shown in Table III the later half of which also shows the effect of delay in adding glucose solution and also of using larger excess of permanganate solution (0·2 – 0·5 ml.).

(*C*) *Determination of small quantities of iodides* :

The iodometric procedure (*B*) with slight modification can be used for determining upto 0·5 mgm of iodine present as iodide.

TABLE III

Temperature 25°C

Iodometric determination of iodide solutions

(5 ml. iodide + 5 gm. $(NH_4)_2$ SO_4 + 2 ml 8N H_2SO_4 titrate $KMnO_4$ to pink + glucose solution + KI 1 gm. titrate with thiosulphate).

10 ml microburette used for thiosulphate solution.

KI present	0·1N $KMnO_4$	Excess $KMnO_4$	Colour	Time of adding $C_6H_{12}O_6$	0·1N Thio-sulphate	KI found	% error
166 mgm	10·1 ml	0·1 ml	faint pink	10 to 15 seconds	10·0 ml.	166 mgm	Nil
166 ,,	10·2 ,,	0·2 ,,	pink	,,	10·05 ,,	166·8 ,,	+0·5
166 ,,	10·3 ,,	0 3 ,,	deeper pink	,,	10 1 ,,	167·7 ,,	+1 0
83 ,,	5·1 ,,	0·1 ,,	faint pink	,,	5·02 ,,	83·3 ,,	+0·4
66·4 ,,	4·1 ,,	0·1 ,,	,,	,,	4·01 ,,	66·6 ,,	+0·3
16·6 ,,	1·15 ,,	0·15 ,,	,,	,,	1·0 ,,	16·6 ,,	nil
Effect of large excess of $KMnO_4$ and delay in adding glucose							
83·0 mgm	5·5 ml	0·5 ml	deeper pink	15 sec.	5·05 ml	83·8 mgm	+1·0
83·0 ,,	,,	,,	,,	30 sec.	5·2 ,,	86·3 ,,	+4·0
83·0 ,,	,,	,,	,,	1 mnt.	5·5 ,,	87·2 ,,	+5·0
83·0 ,,	,,	,,	,,	5 mnt.	5·3 ,,	88·0 ,,	+6·0
83·0 ,,	5·2	0·2	slight pink	15 sec.	5·04 ,,	83·7 ,,	+0·8
83·0 ,,	,,	,,	,,	30 sec.	5·05 ,,	83·8 ,,	+1·0
83·0 ,,	,,	,,	,,	1 mnt.	5·01 ,,	84·6 ,,	+2·0

TABLE IV

Temperature 25°C

Determination of small quantities of idodides

(5 ml. of KI + 5 gm. amm. sulphate + 2 ml. 8N H_2SO_4 + 0·02 N, $KMnO_4$ titrate + 25 ml 2% glucose added in 10 to 15 sec. + KI then titrate with $Na_2S_2O_3$)

KI added	$KMnO_4$ 0 02N	Colour	Thiosulphate 0·01N	KI found	% error
16·6 mgm	5·4 ml.	faint pink	10·05	16·68 mgm	+0·5
6·64 ,,	2·3 ,,	,, ,,	4·00	6·64 ,,	Nil
3·32 ,,	1·1 ,,	,, ,,	2·01	3·33 ,,	+0·4
1·328 ,,	0·6 ,,	,, ,,	0·81	1·344 ,,	+1·2
0·664 ,,	0·3 ,,	,, ,,	0·40	0·664 ,,	Nil

The permanganate solution was diluted to 0 02N, and the thiosulphate solution to 0·01 N. A 5% glucose solution was found to effect the sensitivity of indicator (starch) slighlty, therefore a 2% solution was prepared and about 25 ml of it added in each determination. Five–10 ml. of iodide solution containing 0·5–15 mgm of iodine was taken in a 250 ml. stoppered conical flask and titrated first with permanganate and then with thiosulphate in the same manner as described earlier. For the thiosulphate titration, 2 ml. starch solution was used as indicator and every end-point was checked by back titrating with one drop of 0·01N iodine solution, when a slight blue tinge appeared. If more than one drop of iodine was needed for producing this tinge then the excess of 0·01N, iodine added was deducted from the thiosulphate reading. The result are depicted in Table IV.

Interferences :

Sulphates, phosphates, chlorides and nitrates do not interfere but bromide does. Among the organic acids, the salts of formic, acetic, propionic, butyric and benzoic were found not to interfere, provided they were not present in large quantities. About 0·1 gm. of the sodium salt in 5 ml. of assay solution can be easily tolerated.

Instead of using sulphuric acid for acidifying the iodide solution, hydrochloric acid (3N) or freshly boiled nitric acid (3N) solution can also be used.

(D) Determination of mercuric salts.

As mercuric salts produce an insoluble precipitate of mercuric iodide when treated with iodide solution, mercury was evaluated by reacting with excess of potassium iodide solution and determining the excess iodide by the iodometric method (*B*). Five ml. of a solution of mercuric salt contaiaing 20–100 mgm of mercury was treated with 10 ml. of 0·2N of potassium iodide solution in a 250 ml. glass stoppered conical flask. The mercuric iodide precipitate dissolved in the excess of iodide solution, which was determined by the iodometric method (*B*). In these titrations on oxidation of the iodide by permanganate in presence of 2N sulphauric acid and 5 gm of ammonium sulphate, not only a precipitate of iodine was obtained but also that of mercuric iodide was thrown down. The latter adsorbed the iodine precipitate and did not interfere with the permanganate end-point or the sharpness of the final starch end-point. A blank was carried out with 5 ml of potassium iodide solution. The mercuric content was calculated from the iodide consumed *i.e.*, the difference in the thiosulphate readings.

1 ml of 0·1N thiosulphate = 10 mgm mercury.

(E) Determination of silver.

Five to 10 ml. of silver solution in water or dilute nitric acid was pipetted into a 250 ml flask. If nitric acid were present then the solution was boiled for a few minutes to drive off dissolved oxides of nitrogen and nitrous acid. The solution cooled and treated with 10 ml. of 0·2N potassium iodide solution and the excess of iodide determined iodometrically by the method (*B*). A blank was earried out with 5 ml. of iodide solution.

1 ml. of 0·1N thiosulphate = 10·8 mgm of silver.

(F) Determination of lead.

As lead iodide precipitate was found to react with acidified permanganate it is not possible to determine the excess iodide without filtering off the lead

iodide precipitate. Therefore dry filtration through a No. 42 Whatman filter paper was carried out and an aliquot of the filtrate taken for evaluating the excess iodide used.

Ten ml. of lead solution containing a few ml. of dilute nitric acid (3N freshly boiled) was treated with 10 ml. of 0·2N potassium iodide solution and shaken well. The amount of lead present in the solution taken did not exceed 150 mgm. The mixture was filtered using a dry filter paper, dry funnel and a dry beaker for receiving the filtrate. Ten ml. of the filtrate was taken by pipette for determining the excess iodide by iodometric method (*B*).

$$Pb\,(NO_3)_2 + 2\,KI = PbI_2 + 2\,KNO_3$$

1 ml of 0 1 N thiosulphate = 10·35 mgm of lead

TABLE V

Determination of silver, mercury and lead by precipitation as iodide

Metal	Mgm. added	Mgm found	% error
Silver as $AgNO_3$	54·0	53·8	–0·4
,,	108·0	107·7	–0·3
,,	129 6	129·0	–0 4
Lead as $Pb\,(NO_3)_2$	20·7	20·6	–0·5
,,	51·8	51·6	–0·4
,,	103·6	103·0	–0·6
Mercury as $HgCl_2$	40·1	40·0	–0·3
,,	80·2	79·8	–0·5
,,	100·3	100·1	–0·2

If the metals are dissolved in dilute nitric acid (3N or less) then an aliquot portion should be boiled for a few minutes, cooled and then precipitated with iodide solution.

CONCLUSION

The iodometric procedure (*B*) is recommended for the determination of iodides in presence of chlorides or nitrates, even if present in large excess. For evaluation of silver the present method is more convenient than the Volhard procedure as the starch end-point is more easily detectable than the red tinge against silver chloride precipitate in Volhard titration. The procedure for mercuric salts, specially mercuric chloride in neutral solution is much more accurate than the titration of mercuric solution with potassium iodide solution.

ACKNOWLEDGEMENT

The author wishes to thank Principal, G. R. Inamdar and Dean, Faculty of Science, S. Ghosh for the facilities provided.

REFERENCES

1. Vogel A. I., "Quantitative Inorganic Analysis" 3rd ed., p. 964, *Longman Green & Co. Londan*, (1962).
2. Kolthoff, I. M. and Belcher, R. "Volumetric analysis", Vol. III, p. 61. *Interscience Publishers*, New York, (1957).
3. Garbatschaff, S. W. and Kassatkina, I. A. *Z. anorg. u.allgem. Chem.* 191 ; 104, (1930).

ON AN INTERPRETATION OF DE BROGLIE'S RELATIONS

By

R. L. DASVARMA

Department of Mathematics, L. S. College, Muzaffarpur

[Received on 14th December, 1965]

ABSTRACT

De Broglie's relations, $E = h\nu$, $p = h/\lambda$, connecting the energy and momentum of a mechanical particle on the one hand, and the frequency and wave-length of the associated matter wave on the other, is shown here to be a direct consequence of the atomicity of action. Based on this, a simple interpretation of these relations can be given without introducing the concept of wave-particle dualism.

1. De Broglie's hypothesis, as is well-known, forms the starting point of wave mechanics. The hypothesis may be stated as follows :

The presence of a material particle is connected with the presence of a measurable wave field in space-time. The connection is such that to a particle of momentum $\vec{p}$ and energy E, there corresponds a plane monochromatic wave, travelling in the direction of the momentum vector. The wave vector $\vec{\sigma}$ and the frequency ν are given by

$$\vec{\sigma} = \vec{p}/h, \qquad \nu = E/h, \qquad \ldots . (1)$$

where h is the universal quantum of action[1].

It will be shown that the relations (1) are direct consequence of the atomicity of action and, as a result, can be given a very simple interpretation, without introducing the above hypothesis.

We shall consider here, the case of a particle moving with a constant momentum (the free particle).

2. Let the particle be of mass m and be moving along the axis of x with constant momentum p, starting from the origin at $t = O$. The action function for the motion is

$$S(x, t) = mx^2/2t, \qquad \ldots . (2)$$

with the constant momentum

$$p = mx/t, \qquad \ldots . (3)$$

The function $S(x, t)$ is a function of the two independent variables x and t. Let us consider its change for a change of x alone, t being fixed. If, for a small change λ in x, the corresponding change in S be h, we have, from (2).

$$S = mx^2/2t, \qquad S + h = m(x + \lambda)^2/2t,$$

whence, on subtraction, we get

$$h = \lambda mx/t,$$

neglecting the square of λ. By virtue of (3), we, therefore, have

$$h = \lambda p. \qquad \ldots . (4)$$

3. Similarly, we may consider the change in S due to the change in t only, x remaining fixed. If

$$S(x, t + \tau) = S(x, t) - h,$$

where τ is a small interval of time, we have, from (2),

$$mx^2/2(t + \tau) = mx^2/2t - h,$$

whence on subtraction and neglecting the square and higher powers of τ/t, we get

$$h = \tau mx^2/2t^2.$$

Now, the energy, E, of the particle is given by

$$E = p^2/2m = mx^2/2t^2, \qquad \ldots . (5)$$

from (3). Therefore, we have

$$h = E\tau. \qquad \ldots . (6)$$

4. The relations (4) and (6) or the equivalent relations

$$p = h\,\sigma,\ E = h\nu \qquad \ldots . (7)$$

may, now, be interpreted as follows :

If the change of S takes place in quanta h, and, for a given t, S changes by a quantum in an interval of length λ, then p is the ratio of the change, h, in S to the interval, λ, of length in which the change takes place. The number σ is the frequency of such changes per unit length Similarly, if, for a given x, S changes by a quantum, h, in an interval of time, τ, E is the ratio of h to the interval τ. The number ν is the frequency of such changes per unit time. The momentum and the energy are, thus, the rates of change of action per unit length and per unit time, for a given instant and a given point, respectively. For a free particle both these rates are constant and, therefore, independent of the given instant and the given point, respectively.

5. In the classical Hamiltonian mechanics, the momentum p and the energy E are derived from the action function $S(x, t)$ through the relations

$$p = \frac{\partial S}{\partial x},\ E = -\frac{\partial S}{\partial t}, \qquad \ldots . (8)$$

which show that they are the partial rates of change of the action with respect to coordinate and time, respectively, at a given instant and at a given point. We may, therefore, carry over the relations

$$p = h/\lambda,\ E = h/\tau, \qquad \ldots . (9)$$

to the relations (8) of classical mechanics by passage to the limit, when h, λ and τ tend to zero.

6. De Broglie's relations (9), are, thus, generalisations of the classical relations (8) resulting from the atomicity of action and reduce to the latter only in the limit when the quantum of action h tends to zero. It is to be noted that the negative sign in the second relation in (8) appears because, for a fixed x, S diminishes as t increases. In deducing the corresponding relation in (9), this was taken into consideration so that no negative sign appears here.

REFERENCE

1. Krammers, H. A. Foundations of Quantum Theory, Amsterdam, (1957).

THEOREMS ON $M_{k,m}$ TRANSFORM AND INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

By

S. K. KULSHRESHTHA

Department of Mathematics, Rajasthan University, Jaipur

[Received on 25th September, 1965]

ABSTRACT

The object of this paper is to obtain theorems on $M_{k,m}$ transform. First theorem relates $M_{k,m}$ transform and Banerjee transform and the second theorem relates $M_{k,m}$ transform and generalized Hankel transform. One infinite integral involving hypergeometric functions has also evaluated with the help of this theorem.

INTRODUCTION

The classical Laplace Transform

$$\psi(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt \qquad \ldots . (1{\cdot}1)$$

has been generalized in the form[5]

$$\phi(p) = p\int_0^\infty (2pt)^{-\frac{1}{4}}\, M_{k,m}(2pt)\, h(t)\, dt \qquad \ldots . \quad (1{\cdot}2)$$

The equation (1·2) reduces to (1·1) when $k = -m = \frac{1}{4}$ due to the identity

$$M_{1/4,\,-1/4}(x) = x^{\frac{1}{4}}\; e^{-x/2} \qquad \ldots . \quad (1{\cdot}3)$$

We shall denote the integral equation (1·2) symbolically as

$$\phi\,(p)\,\frac{M}{\overline{\overline{k,\ m}}}\, h(t)$$

and the equation (1·1) as usual shall be denoted as

$$\psi(p) \doteqdot h(t)$$

Banerjee has also generalized (1·1) as

$$\phi\,(p) = p\int_0^\infty (2px)^{m-\mu-1}\; e^{-px(n/2-1)}\; W_{k,\,\mu}(2px)\, h(x)\, dx \quad \ldots . \quad (1{\cdot}4)$$

which we shall symbolically denote as

$$\phi\;(p)\,\frac{B}{\overline{\overline{m,\,n,\,k,\,\mu}}}\; h(t).$$

Also we have, Generalized Hankel Transforms defined as

$$\phi(p) = (\tfrac{1}{2})^{\lambda} \int_0^{\infty} (pt)^{\lambda+1/2} J_{\lambda}^{\mu} (p^2 t^2/_4) \, h(t) \, dt \qquad \ldots . \quad (1\cdot 5)$$

where $J_{\lambda}^{\mu}(x)$ is Maitland's generalized Bessel's Function defined as

$$J_{\lambda}^{\mu}(x) = \sum_{r=0}^{\infty} \left\{ (-x)^r / \lfloor r \cdot \Gamma(1 + \lambda + \mu r) \right\} \qquad \ldots . \quad (1\cdot 6)$$

When $\mu = 1$ (1·6) reduces to Hankel Transform defined as

$$\phi(p) = \int_0^{\infty} (pt)^{1/2} J_{\lambda}(pt) \, h(t) \qquad \ldots . \quad (1\cdot 7)$$

We shall denote (1·5) symbolically as

$$\phi(p) \, \frac{J}{\overline{\lambda, \mu}} \, h(t)$$

and (1·7) as usual shall be denoted as

$$\phi(p) \, \frac{J}{\overline{\lambda}} \, h(t).$$

2. Theorem 1. If

$$\phi(p) \, \frac{M}{\overline{\lambda, \nu}} \, \psi(t)$$

and

$$p^{3/4} \psi(p) \, \frac{B}{\overline{m, n, k, \mu}} \, h(t) \qquad \ldots . \quad (2\cdot 1)$$

then

$$\phi(p) = p^{v} (p_{/2})^{5/1} \int_0^{\infty} t^{-v-3/2} \, h(t) \; F^{(5)} \left[\begin{matrix} 1 + v + m, \; 1 + v + m - 2\mu : \; \tfrac{1}{2} + \lambda + v \\ \tfrac{3}{2} - k + m - \mu + v \qquad\qquad : \; 2v + 1 \end{matrix} \; ; \right.$$

$$\left. \frac{-p}{t}, \, 1 - \frac{n}{4} - \frac{p}{2t} \right] dt \qquad \ldots . \quad (2\cdot 2)$$

provided $R(\tfrac{1}{2} - k + \mu) > 0$, $R(2\mu + 1) > 0$, $R(p) > 0$, n is a positive integer, $R(m + v - \mu) > | R(\mu) | - 1$ the integral involved in (2·2) is absolutely convergent and the generalized double hypergeometric series

$$F^{(5)} \left[\begin{matrix} a, \, b : d \\ c : e \end{matrix} \; ; x, y \right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{r+s} \, (b)_{r+s} \, (d)_r}{(c)_{r+s} \, (e)_r \lfloor r \, \lfloor s} \, x^r \, y^s$$

is a very particular case of the Kampe's de Feriet's hypergeometric function of two variables of higher order and in Kampe de Feriet's notation it is the function

$$F \left[\begin{array}{c|c|c} 2 & c, d & \\ 1 & a, b & x \\ 1 & e & y \\ 1 & a', b' & \end{array} \right]$$

Proof. Using the result [6, p. 437]

$$M_{\lambda,v}\ (2\alpha t) \underset{m,\,n,\,k,\,\mu}{\overset{B}{=\!=}}\ \tfrac{1}{2}\ \alpha^{v+\frac{1}{2}}\ p^{-v-\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(1-n/4+\alpha/2p)^r\ \Gamma(1+v+m-\mu\pm\mu+r)}{\underline{|r}\ \ \Gamma(\frac{3}{2}-k+m+v-\mu+r)} \times$$

$$\times\ {}_3F_2\left[\begin{matrix}\frac{1}{2}+\lambda+v,\ 1+v+m-\mu\pm\mu+r \\ 2v+1,\ \frac{3}{2}-k+m+v-\mu+r\end{matrix};-\alpha/p\right]$$

$$=\tfrac{1}{2}(\alpha/p)^{v+\frac{1}{2}}\,F^{(5)}\left[\begin{matrix}1+v+m-\mu\pm\mu:\ \frac{1}{2}+\lambda+v \\ \frac{3}{2}-k+m+v-\mu\ \ :\ 2v+1\end{matrix};-\frac{\alpha}{p},\ 1-\frac{n}{4}+\frac{\alpha}{2p}\right]$$

where $R(m-\mu+v) > |\,R(\mu)\,|-1,\ R(p)>0,$

and the result (2·1) in Goldstein throrem[4] we get

$$\int_0^{\infty} t^{-\frac{1}{4}}\, M_{\lambda,v}\,(2\alpha t)\ \psi(t)\ dt$$

$$=\tfrac{1}{2}\ \alpha^{v+\frac{1}{2}}\int_0^{\infty} t^{-v-\frac{3}{2}}\ h(t)\ F^{(5)}\left[\begin{matrix}1+v+m-\mu\pm\mu:\ \frac{1}{2}+\lambda+x \\ \frac{3}{2}-k+m+v-\mu\ \ :\ 2v+1\end{matrix};-\frac{\alpha}{t},\ 1-\frac{n}{4}+\frac{\alpha}{2t}\right] dt$$

Multiplying both sides by $\alpha(2\alpha)^{-1/4}$ and finally on replacing α by p we obtain the theorem.

Corollary 1. In the theorem if we take $n=2$ and $m=\mu+3/4$, Banerjee Transform reduces to Whittaker Transform defined by Varma[7] and the reduced form of the theorem is as

If $$\phi(p) \underset{\lambda,\,v}{\overset{M}{=\!=}} \psi(t)$$

and $$p^{3/4}\ \psi(p) \underset{k,\mu}{\overset{W}{=\!=}} h(t)$$

then

$$\phi(p) = p^{v}\ (p/2)^{5/4}\int_0^{\infty} t^{-v-3/2}\ h(t)\ F^{(5)}\left[\begin{matrix}\frac{7}{4}+v\pm\mu:\ \frac{1}{2}+\lambda+v \\ \frac{3}{4}-k+v\ \ :\ 2v+1\end{matrix};-\frac{p}{t}\ ,\ \frac{1}{2}-\frac{p}{2t}\right] dt \quad \ldots\ (2{\cdot}5)$$

where $R(\frac{1}{2}-k+\mu)>0$, $R(2\mu+1)>0$, $R(v+\frac{1}{4}) > |\,R(\mu)\,|-1$, $R(p)>0$ and the integral involved in 2·5) is absolutely convergent.

Corollary 2. In the Corollary 1, taking $k=\pm\mu=\frac{1}{4}$, we get

If $$\phi\ (p) \underset{\lambda,v}{\overset{M}{=\!=}} \psi(t)$$

and $$p^{3/4}\ \psi(p) \doteqdot h(t)$$

then

$$\phi(p) = (2p)^{v} \, (p/_2)^{5/4} \int_0^\infty t^{-3/2} \, (t+p)^{-v} \, {}_2F_1\left[\begin{matrix} v+\frac{3}{2}, \frac{1}{2}+\lambda+v \\ 2v+1 \end{matrix} ; \frac{-2p}{t+p}\right] h(t)\, dt \qquad \ldots. \quad (2\cdot6)$$

provided $R(v+1) > 0$, $R(p) > 0$, and the integral involved in (2·6) is absolutely convergent.

Corollary 3. In the theorem taking $\lambda = -v = \frac{1}{4}$ we get

If $$\phi(p) \doteqdot \psi(t)$$

and $$p^{3/4}\, \psi(p) \underset{m,\, n,\, k,\, \mu}{\overset{B}{\doteqdot}} h(t)$$

then

$$\phi(p) = (\tfrac{1}{2})^{5/4}\, p \int_0^\infty t^{-5/4} \; {}_2F_1 \left[\begin{matrix} \frac{3}{4}+m-\mu+\mu \\ \frac{7}{4}-k-\mu+m \end{matrix} ; 1-\frac{n}{4} \; -\frac{3p}{2t}\right] h(t)\, dt \qquad \ldots. \quad (2\cdot7)$$

provided $R(\frac{1}{2}-k+\mu) > 0$, $R(2\mu+1) > 0$, $R(m-\mu-\frac{1}{4}) > |R|(\mu)|-1$, $R(p) > 0$, n is a positive integer and the integral involved in (2·7) is absolutely convergent.

Corollary 4. In the theorem taking $n = 4$, writing $m+\mu+\frac{1}{2}$ for m and replacing $p/_2$ for p, the Banerjee Transform reduces to Varma Transform of first kind and the theorem reduces to

If $$\phi(p) \underset{\lambda, v}{\overset{M}{\doteqdot}} \psi(t)$$

and $$(p/_2)^{3/4}\, \psi(p/_2) \underset{k, \mu}{\overset{V}{\doteqdot}} h(t)$$

then

$$\phi(p) = p^{v}\, (p/_2)^{5/4} \int_0^\infty t^{-v-3/2} \; F^{(5)} \left[\begin{matrix} \frac{3}{2}+m+v+\mu : \frac{1}{2}+\lambda+v \\ 2-k+m+v : 2v+1 \end{matrix} ; -\frac{p}{t}, -\frac{p}{2t}\right] h(t)\, dt \qquad \ldots. \quad (2\cdot8)$$

provided $R(\frac{1}{2}-k+\mu) > 0$, $R(2\mu+1) > 0$, $R(\frac{1}{2}+m+v) > |R(\mu)|-1$, $(R) > 0$, and the integral involved in (2·8) is absolutely convergent.

Example: In the Corollary 2 if we take

$$\psi(t) = t^{\sigma}\, e^{-bt}\, W_{k,\,\mu}(2\alpha t)$$

then [6, p. 437]

$$\phi(p) = \frac{p^{v+5/4}}{2^{\sigma+\frac{1}{2}}\; \alpha^{v+\sigma+\frac{3}{4}}} \, F^{(5)} \left[\begin{matrix} \frac{7}{4}\pm\mu+v+\sigma : \frac{1}{2}+\lambda+v \\ \frac{3}{4}-k+v+\sigma \;\; : 2v+1 \end{matrix} ; 2\alpha(p+\alpha-b) ; \frac{-p}{\alpha}\right] \qquad \ldots. \quad (2\cdot9)$$

Now [3, p. 216], $p^{3/4}\,\psi(p) = p^{\sigma+\frac{3}{4}}\, e^{-bp}\, W_{k,\,\mu}(2ap)$

$$\doteqdot \frac{(2\alpha)^k\,(t-b-\alpha)^{-\sigma-k-\frac{3}{4}}}{\Gamma(-\sigma-k+\frac{1}{4})}\;{}_2F_1\left[\begin{matrix}\frac{1}{2}-k\pm\mu \\ -\sigma-k+\frac{1}{4}\end{matrix};-\;\frac{t-b-\alpha}{2\alpha}\right]$$
$$= h(t)$$

where $|\arg\alpha| < \pi$ and $R(-\sigma-k-\frac{3}{4}) > 0$.

Now using the value of $h(t)$ in relation (2·6), equating this value of $\phi(p)$ with the value of $\phi(p)$ obtained in relation (2·9) and on adjusting the parameters we get

$$\int_0^\infty t^{-3/2}(t+a)^{-v}\,(t-b)^{\gamma-1}\;{}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};-\frac{t-b}{c}\right]\;{}_2F_1\left[\begin{matrix}v+\frac{3}{2},\lambda+v+\frac{1}{2} \\ 2v+1\end{matrix};-\frac{2a}{t+a}\right]dt$$
$$= \frac{2\sqrt{2}\,\sqrt{(\gamma)}}{c^{2v-\gamma+(\alpha+\beta-1)/2}}\,F^{(6)}\left[\begin{matrix}\frac{1}{2}+v+\alpha-\gamma,\ \frac{1}{2}+v+\beta-\gamma : \frac{1}{2}+\lambda+v \\ \frac{1}{2}+v+\alpha+\beta-\gamma \qquad : 2v+1\end{matrix};\right.$$
$$\left. 2a(a+b-\frac{c}{2}),\ \frac{-2a}{c}\right] \qquad \ldots. \quad (2·9)$$

where $R(\gamma) > 0$, $|\arg c/_2| < \pi$ and $R(a > 0$.

3. Theorem 2. If

$$g(p) \underset{\lambda,\mu}{\overset{J}{=}} e^{-\alpha t} f(t)$$

and
$$f(p) \underset{k,\,m}{\overset{M.}{=}} t^\sigma\,\phi(t)$$

then

$$\psi(p) = 2^{m-\lambda+\frac{1}{4}}\;p^{\lambda+\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-p^2/_4)^r\;\Gamma(^{11}/_4+m+\lambda+2r)}{\lfloor r\;\Gamma(1+\lambda+\mu r)}\times$$
$$\times\int_0^\infty x^{\sigma+m+\frac{1}{4}}(\alpha+x)^{-m-\lambda-2r-11/4}\;{}_2F_1\left[\begin{matrix}\frac{1}{2}-k-m,\ ^{11}/_4+m+\lambda+2r \\ 2m+1\end{matrix};\right.$$
$$\left.\frac{2x}{\alpha+x}\right]\phi(x)\,dx \qquad \ldots. \quad (3·1)$$

provided $R(\rho+\lambda+m+{}^{11}/_4) > 0$, $R(\alpha) > 0$, $R(p) > 0$ and

$$\phi(t) = 0(t^\rho)\text{ for small } t$$
$$= 0(e^{-t\beta})\text{ for large } t,\ R(\beta) > 0.$$

Proof: Since

$$g(p) = (\tfrac{1}{2})^\lambda\int_0^\infty (pt)^{\lambda+\frac{1}{2}}\,J_\lambda^\mu\left(\frac{p^2t^2}{4}\right)e^{-\alpha t}\,f(t)\,dt \qquad \ldots. \quad (3·2)$$

and
$$f(p) = p\int_0^\infty (2pt)^{-\frac{1}{4}}\,M_{k,\,m}(2pt)\,t^\sigma\,\phi(t)\,dt \qquad \ldots. \quad (3·3)$$

Substituting the value of $f(t)$ from (3·3) in (3·2). putting the value of Maitland's Bessel function from (1·6), changing the order of integration and then solving the inner integral with the help of the result [3, p. 215].

$$\int_0^\infty e^{-pt}\, t^{v-1}\, M_{k,\mu}(at)\, dt = \frac{\alpha^{\mu+1/2}\, \Gamma(\mu+v+\frac{1}{2})}{(p+\alpha/_2)^{\mu+v+1/2}}\ {}_2F_1\left[\begin{matrix}\mu+v+\frac{1}{2},\ \frac{1}{2}-k+\mu \\ 2\mu+1\end{matrix}\ ;\ \frac{\alpha}{p+\alpha/_2}\right]$$

where $R(\frac{1}{2}+\mu+v) > 0$ and $R(p) > \frac{1}{2} \mid R(\alpha) \mid > 0$, we get the theorem.

The change of the order of integration is justified by virtue of De la vallee Poussin's theorem [1, p. 504] under the conditions stated with the theorem when the integral involved are absolutely convergent.

Corollary : In the theorem 2, if we take $\mu = 1$, we get the reduced form of the theorem as

If $$g(p) \underset{\lambda}{J}\ e^{-\alpha t} f(t)$$

and $$f(p) \frac{M}{k,\, m}\ t^{\sigma}\, \phi(t)$$

then

$$g(p) = \frac{p^{\lambda+1/2}\, \Gamma(\frac{11}{4}+m+\lambda)}{2^{\lambda-m-\frac{1}{2}}\, \Gamma(1+\lambda)} \int_0^\infty x^{\sigma+m+1}\, (\alpha+x)^{-m-\lambda-11/4}\, \phi(x)$$

$$H_4\left[\frac{11}{4}+m+\lambda,\ \frac{1}{2}-k-m,\ 1+\lambda,\ 2m+1\ ;\ \frac{-p^2}{4(\alpha+x)},\ \frac{2x}{(\alpha+x)}\right] dx \quad \ldots\ldots \quad (3\cdot4)$$

provided $R(^{11}/_4+\lambda+m+\rho) > 0$, $R(\alpha) > 0$, $R(p) > 0$,

$$\phi(t) = 0(\,t^{\rho}\,) \text{ for small } t$$

$$= 0(\,e^{-t\beta}\,) \text{ for large } t,\ R(\beta) > 0,$$

and the double generalized hypergeometric function $H_4\,[\alpha, \beta, \gamma, \delta\,;\, x, y]$ is defined as [2, p. 225]

$$H_4\,[\alpha, \beta, \gamma, \delta\,;\, x, y] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_{2r+s}\, (\beta)_s}{(\gamma)_r\, (\delta)_s \lfloor r \ \lfloor s}\, x^r\, y^s.$$

ACKNOWLEDGEMENT

I am grateful to Dr. K. C. Sharma, Rajasthan University, Jaipur, for his guidance and keen interest during the preparation of this paper.

REFERENCES

1. Bromwich, T. J. I'A. 'An introduction to theory of infinite series', (1931).
2. Erdelyi, A. 'Higher Transcedental Functions', Vol. I, *McGraw Hill, New York*, (1953).
3. Erdelyi, A. Tables of integral Transforms', Vol. I, *McGraw Hill, New York*, (1954).
4. Goldstein, S. *Proc. Lond. Math. Soc.*, **34** (2) : 103–125, (1932).
5. Kulshreshtha, S. K. 'On an integral Transform', communicated to *Proc Nat. Acad. Sc.*, India.
6. Srinivasan. 'On the generalized Meijer Transform', *Bull. de L'Academic Polonaise de Sciences*, **11** (7) : 431–440, (1963).
7. Varma, R. S. *Current Sciences*, **15** : 16–17, (1947).

INTEGRALS INVOLVING PRODUCTS OF HYPERGEOMETRIC FUNCTION

By

H. B. MALOO

Department of Mathematics, Jodhpur University, Jodhpur, India

[Received on 29th November, 1965]

INTRODUCTION

The object of the present note is to evaluate some infinite integrals involving products of Hypergeometric functions by the method of operational calculus and as usual, the notation

$$\phi(p) \doteqdot f(t), \text{ stands for } \phi(p) = p\int_0^\infty e^{-pt} f(t)\, dt.$$

Parseval's formula is that if $\phi(p) \doteqdot f(t)$ and $\psi(p) \doteqdot g(t)$, then

$$\int_0^\infty \phi(t)\, g(t)\, \frac{dt}{t} = \int_0^\infty \psi(t)\, f(t)\, \frac{dt}{t}. \qquad \ldots. \quad (1)$$

We take [1, p. 216 (16)]

$$\begin{aligned}\phi(p) &= p\,(p+b)^{-\nu-\mu-\frac{1}{2}}\, {}_2F_1\left(\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ \nu+\lambda+1;\ \frac{p}{p+b}\right)\\ &\doteqdot \frac{\Gamma(\nu+\lambda+1)\, b^{-\mu-\frac{1}{2}}}{\Gamma(\nu+\mu+\frac{1}{2})\,\Gamma(\nu-\mu+\frac{1}{2})}\, t^{\nu-1} e^{-\frac{1}{2}bt}\, W_{-\lambda,\,\mu}(bt)\\ &= f(t),\ R(p+b) > 0,\ R(\nu\pm\mu+\tfrac{1}{2}) > 0 \qquad \ldots. \quad (2)\end{aligned}$$

and [1, p. 213, (8)]

$$\begin{aligned}\psi(p) &= \Gamma(1-k-\lambda)\,(ab)^{-\mu-\frac{1}{2}}\, e^{\frac{1}{2}(a+b)p}\, W_{k,\,\mu}(ap)\, W_{\lambda,\,\mu}(bp)\\ &\doteqdot t^{-k-\lambda}\,(a+t)^{k-\mu-\frac{1}{2}}\,(b+t)^{\lambda-\mu-\frac{1}{2}} \times\\ &\quad {}_2F_1\left(\tfrac{1}{2}-k+\mu,\ \tfrac{1}{4}-\lambda+\mu;\ 1-k-\lambda;\ \frac{t(t+a+b)}{(t+a)(t+b)}\right) \qquad \ldots \quad (3)\\ &= g(t),\ R(1-k-\lambda) > 0,\ |\arg a| < \pi,\ |\arg b| < \pi.\end{aligned}$$

Using these relations (2) and (3) in (1), we get

$$\begin{aligned}&\int_0^\infty t^{-k-\lambda}\,(a+t)^{k-\mu-\frac{1}{2}}\,(b+t)^{\lambda-2\mu-\nu-1}\, {}_2F_1\left(\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ \nu+\lambda+1;\ \frac{t}{b+t}\right)\\ &\qquad \times {}_2F_1\left(\tfrac{1}{2}-k+\mu,\ 1-\lambda+\mu;\ 1-k-\lambda;\ \frac{t(a+b+t)}{(a+t)(b+t)}\right) dt\\ &= \frac{\Gamma(1-k-\lambda)\,\Gamma(\nu+\lambda+1)}{\Gamma(\nu\pm\mu+\frac{1}{2})\, b^{1+2\mu}}\, a^{-\mu-\frac{1}{2}} \int_0^\infty t^{\nu-2}\, e^{\frac{1}{2}at}\, W_{k,\,\mu}(at)\, W_{\lambda,\mu}(bt)\, W_{-\lambda,\mu}(bt)\, dt.\end{aligned}$$

Evaluate the integral on the right with the help of a known result [2, p. 422] on using

$$e^{\frac{1}{2}x}\, W_{k,\,m}(x) = \frac{2^{-k-1}\ \pi^{-3/2}}{\Gamma(\frac{1}{2}+m-k)} \times$$

and

$$G^{4\,2}_{2\,4}\left(\frac{x^2}{4}\middle|\begin{matrix}\frac{1}{2}+\frac{1}{2}k,\ 1+\frac{1}{2}k\\ \frac{1}{4}+\frac{1}{2}m,\ \frac{3}{4}+\frac{1}{2}m,\ \frac{1}{4}-\frac{1}{2}m,\ \frac{3}{4}-\frac{1}{2}m\end{matrix}\right) \quad \ldots . \quad (4)$$

$$W_{\lambda,\,m}(x)\ W_{-\lambda,\,m}(x) = \pi^{-\frac{1}{2}}\, G^{4\,0}_{2\,4}\left(\frac{x^2}{4}\middle|\begin{matrix}1+\lambda,\ 1-\lambda\\ \frac{1}{2},\ 1,\ \frac{1}{2}+m,\ \frac{1}{2}-m\end{matrix}\right); \quad \ldots . \quad (5)$$

we get

$$\int_0^\infty t^{-k-\lambda}(a+t)^{k-\mu-\frac{1}{2}}(b+t)^{\lambda-2\mu-\nu-1}\ {}_2F_1\left(\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ \nu+\lambda+1;\ \frac{t}{b+t}\right)$$

$$\times\ {}_2F_1\left\{\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}-\lambda+\mu;\ 1-k-\lambda;\ \frac{t(a+b+t)}{(a+t)(b+t)}\right\} dt$$

$$= \frac{\Gamma(1-k-\lambda)\ \Gamma(\nu+\lambda+1)\,a^{\frac{1}{2}-\nu-\mu}}{2^{\nu+k+3}\ \pi^2\ \Gamma(\nu\pm\mu+\frac{1}{2})\ b^{1+}} \times$$

$$G^{6\,4}_{6\,6}\left(\frac{b^2}{a^2}\middle|\begin{matrix}\frac{5}{4}-\frac{1}{2}\mu-\frac{1}{2}\nu,\ \frac{3}{4}-\frac{1}{2}\mu-\frac{1}{2}\nu,\ \frac{5}{4}+\frac{1}{2}\mu-\frac{1}{2}\nu,\ \frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}\nu,\ 1+\lambda,\ 1-\lambda\\ 1-\frac{1}{2}\nu-\frac{1}{2}k,\ \frac{1}{2}-\frac{1}{2}k-\frac{1}{2}\nu,\ \frac{1}{2},\ 1,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu\end{matrix}\right), \quad \ldots . \quad (6)$$

for $R(1-k-\lambda)>0$, $R(-2\mu)>0$, $R(3\mu+\nu+\frac{1}{2})>0$, $|\arg a|<\pi$, $|\arg b|<\pi$.

Again we take [1, p- 215] (11)]

$$\phi(p) = p\,(p+b)^{-\nu-\mu-\frac{1}{2}}\ {}_2F_1\left(\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ 1+\tfrac{1}{2}\mu;\ \frac{b}{b+p}\right)$$

$$\doteqdot \frac{b^{-\mu-\frac{1}{2}}}{\Gamma(\frac{1}{2}+\mu+\nu)}\ t^{\nu-1}\,e^{-\frac{1}{2}bt}\ M_{-\lambda,\,\mu}(bt) \quad \ldots . \quad (7)$$

$$= f(t),\ R(\mu+\nu+\tfrac{1}{2})>0,\ R(p)>0.$$

Using the relations (3) and (7) in (1), we get

$$\int_0^\infty t^{-k-\lambda}\ (a+t)^{k-\mu-\frac{1}{2}}\ (b+t)^{\lambda-2\nu-1-\nu}\ {}_2F_1\left(\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ 1+2\mu;\ \frac{b}{b+t}\right)$$

$$\times\ {}_2F_1\left\{\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}-\lambda+\mu;\ 1-k-\lambda;\ \frac{t(a+b+t)}{(a+t)\ (b+t)}\right\} dt$$

$$= \frac{\Gamma(1-k-\lambda)\ a^{-\mu-\frac{1}{2}}}{\Gamma(\mu+\nu+\frac{1}{2})\ b^{1+2\mu}}\int_0^\infty t^{\nu-2}\,e^{\frac{1}{2}at}\ W_{k,\,\mu}(at)W_{\lambda,\mu}(bt)M_{-\lambda,\,\mu}(bt)\ dt.$$

Evalute the integral on the right with the help of a known result [2, p. 422] on using

$$W_{\lambda,\,\mu}(x)\ M_{-\lambda,\,\mu}(x) = \frac{\pi^{-\frac{1}{2}}\ \Gamma(1+2\mu)}{\Gamma(\frac{1}{2}-\lambda+\mu)}\ G^{3\,1}_{2\,4}\left(\frac{x^2}{4}\middle|\begin{matrix}1+\lambda,\ 1-\lambda\\ \frac{1}{2},\ 1,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu\end{matrix}\right) \quad \ldots . \quad (8)$$

and (4), we get

$$\int_0^\infty t^{-k-\lambda}(a+t)^{k-\mu-\frac{1}{2}}(b+t)^{\lambda-2\mu-\nu-1}\,{}_2F_1\left(\mu+\nu+\tfrac{1}{2},\mu+\lambda+\tfrac{1}{2};\frac{b}{b+t}\right)$$

$$\times\,{}_2F_1\left(\tfrac{1}{2}-k+\mu,\tfrac{1}{2}-\lambda+\mu;1-k-\lambda;\frac{t(a+b+t)}{(a+t)(b+t)}\right)dt$$

$$=\frac{2^{\nu-k-3}\,a^{\frac{3}{2}-\mu-\nu}\,\Gamma(1-k-\lambda)\,\Gamma(1+2\mu)}{\pi^2\,\Gamma(\mu+\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}-\lambda+\mu)\,\Gamma(\frac{1}{2}\pm\mu-k)}$$

$$G^{5\,5}_{6\,6}\left(\frac{b^2}{a^2}\,\middle|\,\begin{matrix}\frac{5}{4}-\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{3}{4}-\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{5}{4}-\frac{1}{2}\nu+\frac{1}{2}\mu,\frac{3}{4}-\frac{1}{2}\nu+\frac{1}{2}\mu,1+\lambda,1-\lambda\\1-\frac{1}{2}k-\frac{1}{2}\nu,\frac{1}{2}-\frac{1}{2}k-\frac{1}{2}\nu,\frac{1}{2},1,\frac{1}{2}+\mu,\frac{1}{2}-\mu\end{matrix}\right),\quad\ldots\quad(9)$$

for $R(1-k-\lambda)>0$, $R(\frac{1}{2}+3\mu+\nu)>0$, $R(-2\mu)>0$, $R(-\nu-\lambda)>0$, $|\arg a|<\pi$, $|\arg b|<\pi$.

Similarly we take [1, p. 216]

$$\phi(p)=p(p+a+b)^{-\nu-\mu-\frac{1}{2}}\,{}_2F_1\left\{\mu+\nu+\tfrac{1}{2},\mu+\lambda+\tfrac{1}{2};\nu+\lambda+1;\frac{(p+a)}{(p+a+b)}\right\}$$

$$\doteqdot\frac{\Gamma(\nu+\lambda+1)\,b^{-\mu-\frac{1}{2}}}{\Gamma(\frac{1}{2}\pm\mu+\nu)}\,t^{\nu-1}\,e^{-(a+\frac{1}{2}b)t}\,W_{-\lambda,\mu}(bt)\quad\ldots\quad(10)$$

$$=f(t),\ R(\nu\pm\mu+\tfrac{1}{2})>0,\ R(p+a+b)>0.$$

Using the relations (3) and (10) in (1), we get

$$\int_0^\infty t^{-k-\lambda}(a+t)^{k-\mu-\frac{1}{2}}(b+t)^{\lambda-\mu-\frac{1}{2}}(a+b+t)^{-\nu-\mu-\frac{1}{2}}$$

$${}_2F_1\left\{\mu+\nu+\tfrac{1}{2},\mu+\lambda+\tfrac{1}{2};\nu+\lambda+1;\frac{(a+t)}{(a+b+t)}\right\}$$

$$\times\,{}_2F_1\left(\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}-\lambda+\mu;\ 1-k-\lambda;\frac{t(a+b+t)}{(a+t)(b+t)}\right)dt$$

$$=\frac{a^{-\mu-\frac{1}{2}}\,\Gamma(1-k-\lambda)\,\Gamma(\nu+\lambda+1)}{b^{1+2\mu}\,\Gamma(\nu\pm\mu+\frac{1}{2})}\int_0^\infty t^{\nu-2}\,e^{-\frac{1}{2}at}\,W_{k,\mu}(at)\,W_{\lambda,\mu}(bt)\,W_{-\lambda,\mu}(bt)\,dt.$$

Evaluate the integral on the right with the help of a known result [2, p. 422] on using

$$e^{-\frac{1}{2}x}\,W_{k,\mu}(x)=\frac{2k}{\pi^{\frac{1}{2}}}\,G^{4\,0}_{2\,4}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2}-\frac{1}{2}k,1-\frac{1}{2}k\\\frac{3}{4}+\frac{1}{2}\mu,\frac{3}{4}-\frac{1}{2}\mu,\frac{1}{4}+\frac{1}{2}\mu,\frac{1}{4}-\frac{1}{2}\mu\end{matrix}\right)\quad\ldots\quad(11)$$

and (5), we get

$$\int_0^\infty t^{-k-\lambda}(a+t)^{k-\mu-\frac{1}{2}}(b+t)^{\lambda-\mu-\frac{1}{2}}(a+b+t)^{-\nu-\mu-\frac{1}{2}}$$

$${}_2F_1\left\{\nu+\mu+\tfrac{1}{2},\mu+\lambda+\tfrac{1}{2};\nu+\lambda+1;\frac{(a+t)}{(a+b+t)}\right\}$$

$$\times\,{}_2F_1\left\{\tfrac{1}{2}-k+\mu,\tfrac{1}{2}-\lambda+\mu;1-k-\lambda;\frac{t(a+b+t)}{(a+t)(b+t)}\right\}dt$$

$$=\frac{2^{\nu+k-4}\,a^{-\nu-\mu-\frac{5}{2}}\,\Gamma(1-k-\lambda)\,\Gamma(\nu+\lambda+1)}{\pi\,\Gamma(\nu\pm\mu+\frac{1}{2})\,b^{1+2\mu}}$$

$$G^{4\,4}_{6\,6}\left(\frac{b^2}{a^2}\,\middle|\,\begin{matrix}\frac{3}{4}-\frac{\nu}{2}-\frac{1}{2}\mu,\ \frac{3}{4}-\frac{1}{2}\nu+\frac{1}{2}\mu,\ \frac{5}{4}-\frac{1}{2}\nu-\frac{1}{2}\mu,\ \frac{5}{4}-\frac{1}{2}\nu+\frac{1}{2}\mu,\ 1+\lambda,\ 1-\lambda\\ \frac{1}{2},\ 1,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ 1+\frac{1}{2}k-\frac{\nu}{2},\ \frac{1}{2}+\frac{1}{2}k-\frac{\nu}{2}\end{matrix}\right) \quad \ldots\ldots (12)$$

for $R(1-k-\lambda)>0$, $R(3\mu+\nu+\frac{1}{2})>0$, $R(-2\mu)>0$, $|\arg a|<\pi$, $|\arg b|<\pi$.

Similarly again we take [1, p. 215]

$$\phi(p)=p\,(p+a+b)^{-\nu-\mu-\frac{1}{2}}\,{}_2F_1\left(\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ 1+2\mu;\ \frac{b}{p+a+b}\right)$$

$$\doteqdot \frac{b^{-\mu-\frac{1}{2}}}{\Gamma(\mu+\nu+\frac{1}{2})}\,t^{\nu-1}e^{-(a+\frac{1}{2}b)t}\,M_{-\lambda,\,\mu}(bt) \quad \ldots\ldots (13)$$

$$= f(t),\ R(\mu+\nu+\tfrac{1}{2})>0,\ R(p+b)>0.$$

Using the relations (3) and (13) in (1), we get

$$\int_0^\infty t^{-k-\lambda}(a+t)^{k-\mu-\frac{1}{2}}(b+t)^{\lambda-\mu-\frac{1}{2}}(a+b+t)^{-\nu-\mu-\frac{1}{2}}$$

$${}_2F_1\left\{\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+\tfrac{1}{2};\ 1+2\mu;\ \frac{b}{(a+b+t)}\right\}$$

$$\times\ {}_2F_1\left\{\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}-\lambda+\mu;\ 1-k-\lambda;\ \frac{t(a+b+t)}{(a+t)\,(b+t)}\right\}dt$$

$$=\frac{a^{-\mu-\frac{1}{2}}\,\Gamma(1-k-\lambda)}{b^{1+2\mu}\,\Gamma(\mu+\nu+\frac{1}{2})}\int_0^\infty t^{\nu-2}\,e^{-\frac{1}{2}at}\,W_{k,\mu}(at)\,W_{\lambda,\mu}(bt)\,M_{-\lambda,\mu}(bt)\,dt.$$

Evaluate the integral with the help of a known result [2, p. 422], on using (11) and (8) we get

$$\int_0^\infty t^{-k-\lambda}(a+t)^{k-\mu-\frac{1}{2}}(b+t)^{\lambda-\mu-\frac{1}{2}}(a+b+t)^{-\nu-\mu-\frac{1}{2}}$$

$${}_2F_1\left\{\mu+\nu+\tfrac{1}{2},\ \mu+\lambda+1;\ 1+2\mu;\ \frac{b}{(a+b+t)}\right\}$$

$$\times\ {}_2F_1\left\{\tfrac{1}{2}-k+\mu,\ \tfrac{1}{2}-\lambda+\mu;\ 1-k-\lambda;\ \frac{t(a+b+t)}{(a+t)\,(b+t)}\right\}dt$$

$$=\frac{2^{k+\nu-2}\,a^{\frac{3}{2}-\nu-\mu}\,\Gamma(1-k-\lambda)\,\Gamma(1+2\mu)}{\pi\,\Gamma(\mu+\nu+\frac{1}{2})\,\Gamma(\frac{1}{2}-\lambda+\mu)}$$

$$G^{3\,5}_{6\,6}\left(\frac{b^2}{a^2}\,\middle|\,\begin{matrix}\frac{3}{4}-\frac{1}{2}\nu-\frac{1}{2}\mu,\ \frac{3}{4}-\frac{1}{2}\nu+\frac{1}{2}\mu,\ \frac{5}{4}-\frac{1}{2}\mu-\frac{1}{2}\nu,\ \frac{5}{4}-\frac{1}{2}\nu+\frac{1}{2}\mu,\ 1+\lambda,\ 1-\lambda\\ \frac{1}{2},\ 1,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu,\ 1-\frac{\nu}{2}+\frac{1}{2}k,\ \frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}k\end{matrix}\right) \quad \ldots\ldots (14)$$

for $R(1-k-\lambda)>0$, $R(3\mu+\nu+\frac{1}{2})>0$, $R(-2\mu)>0$, $|\arg a|<\pi$, $|\arg b|<\pi$.

REFERENCES

1. Erdelyi, A. *et al.* Tables of Integral transforms, Vol. I, McGraw Hill New York (1954).
2. Erdelyi, A. *et al.* Tables of Integral transform, Vol. II, McGraw Hill New York (1954),

A GENERALISED THEOREM ON HANKEL AND VARMA TRANSFORMS

By

G. K. GOYAL *and* A. N. GOYAL

Department of Mathematics, University of Rajasthan, Jaipur

[Received on 25th October, 1965]

ABSTRACT

Integrals involving special functions have been evaluated with the help of a generalised theorem on Hankel and Varma Transforms.

INTRODUCTION

The Hankel and Varma transforms defined as

$$\phi(p) = p \int_0^\infty (pt)^{\frac{1}{2}} J_\nu (pt) f(t)\, dt,\ R(p) > 0$$

and

$$\psi(p) = p \int_0^\infty (pt)^{\beta-\frac{1}{2}}\, e^{-pt/2}\, W_{\alpha,\beta}\, (pt)\, h(t)\, dt,\ R(p) > 0$$

will be represented by $\phi(p) \underset{\nu}{\overset{J}{=}} f(t)$, $\psi(p) \underset{\alpha,\beta}{\overset{V}{=}} h(t)$ hereafter.

Theorem :

If

$$\phi(p) \underset{\mu}{\overset{J}{=}} f(t) \qquad \ldots\ldots (1 \cdot 0)$$

and

$$\psi(p) \underset{\alpha\ \beta}{\overset{V}{=}} t^\lambda\, \phi(t^{r/s}) \qquad \ldots\ldots (1 \cdot 1)$$

(r, s being integers) then

$$\frac{p^\lambda\, (2\pi)^{-\frac{1}{2}}}{(2r)^{\lambda+\alpha+\beta}\, (2s)^{\frac{1}{2}}}\, \psi\,(p)$$

$$= \int_0^\infty G^{s,\,4r}_{4r,\,2s+2r} \left\{ \left(\frac{2r}{p}\right)^{2r} \left(\frac{t}{2s}\right)^{2s} \middle| \begin{matrix} \Delta(-\lambda-2\beta,\, 2r),\ \Delta(-\lambda,\, 2r) \\ \Delta(\tfrac{1}{4}+\mu/_2,\, s),\ \Delta(\alpha-\beta-\lambda-\tfrac{1}{2},\, 2r), \end{matrix} \right\} f(t)\, dt$$

$$\Delta(\tfrac{1}{4}-\mu/_2,\, s) \ldots \quad (1 \cdot 2)$$

where $R(p) > 0$, $R(\lambda+\alpha+\beta+r/_s+\frac{1}{2}) < 0$, $R(\lambda+\mu r/_s+3r/_s+1) > 0$, $R(\lambda+\mu r/_s+3r/_{2s}+2\beta+1) > 0$

and the symbol $\Delta(\xi \pm \eta,\, \delta)$ stands for the series

$$\frac{\xi+\eta}{\delta},\ \frac{\xi+\eta+1}{\delta},\ \ldots\ldots\ \frac{\xi+\eta+\delta-1}{\delta},\ \frac{\xi-\eta}{\delta},\ \frac{\xi-\eta+1}{\delta},\ \ldots\ldots\ \frac{\xi-\eta+\delta-1}{\delta};$$

Proof: Substitute the value of $\phi(t^{r/s})$ in $\psi(p)$, change the order of integration, which is justified under the conditions stated with the theorem, evaluate the t-integral with the help of Saxena's result (1960, 4).

Simplify with the help of Erdelyi (3, p. 209) to obtain equation (1·2).

Corollary 1. (*a*) Write $r = s = 1$ in equations (1·1), (1·2), then (1·2) gives with appropriate changes in (1·1)

$$\frac{\pi p^{-2\beta}\, 2^{-\lambda-2\alpha}\, \Gamma(1-2\beta)}{\Gamma\left(\frac{3}{4}-\frac{\lambda+\mu+2\beta}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\lambda-\mu+2\beta}{2}\right)\Gamma(\frac{3}{2}-\alpha-\beta)}\,\psi(p)$$

$$= \int_0^\infty t^{-\lambda-2\beta}\, {}_4F_3\left\{\begin{matrix}\frac{3}{4}-\frac{\lambda+\mu+2\beta}{2}, \frac{5}{4}-\frac{\alpha+\beta}{2}, \frac{3}{4}-\frac{\alpha+\beta}{2}, \frac{3}{4}-\frac{\lambda-\mu+2\beta}{2}; \frac{-t^2}{p^2}\\ \frac{1}{2}, 1-\beta, \frac{1}{2}-\beta\end{matrix}\right\} f(t)\, dt \qquad \ldots. \quad (1{\cdot}3)$$

where $R(p) > 0$, $R(\lambda+\alpha+\beta+3/2) < 0$, $R(\lambda+\mu+5/2) > 0$, $R(\lambda+\mu+2\beta+5/2) > 0$.

(*b*) Write $r = s = 1$ and $\alpha + \beta = \frac{1}{2}$ in equations (1·1), (1·2) then with appropriate changes in (1·1), (1·2) gives

$$\frac{\pi\, p^{2\alpha-1}\, 2^{\frac{1}{2}-2\alpha-\lambda}\, \Gamma(2\alpha)}{\Gamma\left(\frac{1}{4}-\frac{\lambda+\mu}{2}+\alpha\right)\Gamma\left(\frac{1}{4}-\frac{\lambda-\mu}{2}+\alpha\right)}\psi(p)$$

$$= \int_0^\infty t^{2\alpha-\lambda-1}\, {}_3F_2\left\{\begin{matrix}\frac{1}{4}+\alpha-\frac{\lambda+\mu}{2}, 1, \frac{1}{4}+\alpha-\frac{\lambda-\mu}{2}; \frac{-t^2}{p^2}\\ \frac{1}{2}+\alpha, \alpha\end{matrix}\right\} f(t)\, dt \qquad \ldots. \quad (1{\cdot}4)$$

where $R(p) > 0$, $R(\lambda + 2) < 0$, $R(\lambda + \mu + 5/2) > 0$, $R(\alpha) > \frac{1}{2}$.

(*c*) Write $r = s = 1$, $\beta = 0$, $\alpha = \frac{1}{2}$, $\lambda + \mu = \frac{1}{2}$ in (1·1), (1·2) then with appropriate changes in (1·1), (1·2) will give

$$\frac{2^{\mu-1}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\mu)}\,\psi(p) = \int_0^\infty t^{\mu-\frac{1}{2}}\, {}_1F_1\left(\frac{1}{2}+\mu; \frac{1}{2}; \frac{-t^2}{p^2}\right) f(t)\, dt \qquad \ldots. \quad (1{\cdot}5)$$

where $R(p) > 0$, $R(\mu) > 5/2$.

Corollary 2. (*a*) Write $r = 1$, $s = 2$, $\lambda = \rho - \beta - 3/4$ in (1·1), (1·2) then with appropriate changes in (1·1), (1·2) gives

$$\frac{2^{\mu}\, p^{\rho-\beta+\frac{\mu}{2}}\, \Gamma(\mu+1)\, \Gamma(\frac{3}{2}-\alpha+\frac{\mu}{2}+\rho)}{\Gamma\left(1+\beta+\rho+\frac{\mu}{2}\right)\Gamma\left(1-\beta+\rho+\frac{\mu}{2}\right)}\,\psi(p)$$

$$= \int_0^\infty t^{\mu+\frac{1}{2}} {}_2F_2 \left\{ \begin{matrix} 1+\rho+\frac{\mu}{2}+\beta,\ 1+\rho+\frac{\mu}{2}-\beta \\ \mu+1,\ \frac{3}{2}-\alpha+\rho+\frac{\mu}{2} \end{matrix} ; \frac{-t^2}{4p} \right\} f(t)\, dt \qquad \ldots . (1{\cdot}6)$$

where $R(p) > 0$, $R(\rho + \alpha) < \frac{1}{4}$, $R(\rho \pm \beta + \frac{\mu}{2} + 1) > 0$.

(*b*) Write $\mu = \beta$, $\rho = -1 - \beta/_2$ in (1·6) we get

$$\frac{\Gamma(\frac{3}{2}-\alpha)}{\Gamma(1-\beta)} \left(\frac{2}{p}\right)^\beta \psi(p) = \int_0^\infty t^{+\beta} {}_1F_1 \left(1-\beta\, ; \tfrac{3}{2}-\alpha\, ; \frac{-t^2}{4p}\right) f(t)\, dt \qquad \ldots . (1{\cdot}7)$$

where $R(p) > 0$, $R(2\alpha - \beta) < 5/_2$, $R(\beta) > 0$.

Example 1. Let $f(t) = t^\rho \ e^{-a^2 t^2/4} D_\nu(at)$
then (2, p. 79) gives

$$\phi(p) = \frac{2^{-c-3\mu/2}\, \pi^{\frac{1}{2}}\, \Gamma(2b)\, p^{\mu+3/2}}{a^{2b}\, \Gamma(b)\, \Gamma(b+\frac{1}{2})} E\left(b,\ b+\tfrac{1}{2}\, ; \mu+1,\ c+\tfrac{1}{2}\, ; 2a^2/p^2\right)$$

where $|\arg a| < \pi/_4$, $R(\rho+\mu) > -3/_2$, $2b = \rho + \mu + 3/_2$, $2c = \rho - \nu + 3/_2$ and (2, p. 416) gives after some simplification

$$\psi(p) = \frac{m^{\lambda+\beta+\alpha+\mu+3/2}\ p^{-\lambda-\mu-3/2}\ \Gamma(2b)}{2^{c+\frac{3\mu}{2}+\frac{1}{2}}\ a^{2b}\ \Gamma(b)\ \Gamma(b+\frac{1}{2})} E(b,\ b+\tfrac{1}{2}\, ; \mu+1,\ c+\tfrac{1}{2}\, ; 2a^2p^2/m^2)$$

where $R(\lambda + \beta + \mu + 2) > |R(\nu)| - \frac{1}{2}$, $m = 1, 2, 3, \ldots$

Using (13) we get

$$\frac{\pi\ \Gamma(1-2\beta)\ \Gamma(2b)\ \Gamma(b)\ \Gamma(b+1)\ m^{\lambda+\alpha+\beta+\mu+3/2}\, p^{\lambda+\mu+3/2-2\beta}}{\Gamma\left(\frac{3}{4}-\frac{\lambda+\nu+2\beta}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\lambda-\nu+2\beta}{2}\right)\Gamma(\frac{3}{2}-\alpha-\beta)\ 2^{c+3\mu/2+\lambda+2\alpha}\ a^{2b}}$$

$$\times\ E(b,\ b+\tfrac{1}{2}\, ; \mu+1\, ; c+\tfrac{1}{2}\, ; 2a^2\, p^2/m^2)$$

$$= \int_0^\infty t^{\rho-\lambda-2\beta} {}_4F_3 \left\{ \begin{matrix} \frac{3}{4}-\frac{\lambda+\nu+2\beta}{2},\ \frac{5}{4}-\frac{\alpha+\beta}{2},\ \frac{3}{4}-\frac{\alpha+\beta}{2},\ \frac{3}{4}-\frac{\lambda-\nu+2\beta}{2} \\ \frac{1}{2},\ 1-\beta,\ \frac{1}{2}-\beta \end{matrix} ; \frac{-t^2}{p^2} \right\}$$

$$e^{-\frac{1}{4}a^2t^2}\, D_\nu(at)\, dt$$

where $R(\rho + \alpha + \nu - \beta - \lambda) < \frac{1}{2}$, $R(p) > 0$, $R(\rho) < \frac{1}{2}$.

Example 2. Let

$$f(t) = 2\sqrt{t}\, J_\mu(\sqrt{2at})\, K_\mu(\sqrt{2at})$$

then (2, p. 30) gives

$$\phi(p) = p^{-\frac{1}{2}}\, e^{-a/p}$$

where $R(a) > 0$, $R(p) > 0$ and (2, p. 412)

$$\psi(p) = p^{\beta+\frac{1}{2}} a^{\rho+\frac{\mu}{2}-1} G^{4\,0}_{2\,4}\left(ap \,\middle|\, \begin{matrix} 1-\alpha, \frac{3}{2}-\mu-\rho \\ \frac{1}{2}+\beta, \frac{1}{2}-\beta, \frac{3}{2}+\frac{\mu}{2}-\rho, \frac{1}{2}-\frac{3\mu}{2}-\rho \end{matrix} \right)$$

where $R(a) > 0$, $R(p) > 0$.

Now using (1·6) we get

$$\frac{\Gamma(\mu+1)\ \Gamma\left(\frac{3}{2}-\alpha+\frac{\mu}{2}+\rho\right) 2^{\mu-1}\, p^{\rho+\frac{\mu}{2}+\frac{1}{2}}}{\Gamma\left(1+\beta+\frac{\mu}{2}+\rho\right)\ \Gamma\left(1-\beta+\frac{\mu}{2}+\rho\right) a^{1-\rho-\frac{\mu}{2}}}$$

$$G^{4\,0}_{2\,4}\left(ap \,\middle|\, \begin{matrix} 1-\alpha, \frac{3}{2}-\mu-\rho \\ \frac{1}{2}+\beta, \frac{1}{2}-\beta, \frac{3}{2}+\frac{\mu}{2}-\rho, \frac{1}{2}-\frac{3\mu}{2}-\rho \end{matrix} \right)$$

$$= \int_0^\infty t^{\mu+1} J_\mu(\sqrt{2at})\, K_\mu(\sqrt{2at})\, {}_2F_2 \left\{ \begin{matrix} \lambda+\beta, \lambda-\beta \\ \mu+1, \ -\alpha+\lambda \end{matrix} ; \frac{-t^2}{4p} \right\} dt \quad \ldots \quad (1\cdot 9)$$

where $R(\alpha+\beta+\frac{1}{2}) < 0$, $R(\mu+\frac{1}{2}) > 0$.

Now write $\alpha = \rho$, $\beta = 2\rho + \frac{1}{2}$, $\mu = 2\rho$ in (1·9) to obtain

$$\Gamma(1+2\rho)\ \Gamma(\tfrac{3}{2}-\rho)\, 2^{2\rho}\, \pi^{\frac{1}{2}}\, e^{-\sqrt{ap}} \left(\frac{p}{a}\right)^{\frac{3}{4}} W_{2\rho-\frac{1}{2},\, 4\rho+1}(2\sqrt{ap})$$

$$= \int_0^\infty t^{2\rho+1} J_{2\rho}(\sqrt{2at})\, K_{2\rho}(\sqrt{2at})\, {}_1F_1\left(-2\rho\, ;\ 1-\rho\, ; \frac{-t^2}{4p}\right) dt \quad \ldots . \quad (2\cdot 0)$$

where $R(\rho+\frac{1}{3}) < 0$, $R(p) > 0$, $R(a) > 0$.

REFERENCES

1. Erdelyi, A. Table of Integral transforms Vol. II, (1954).
2. Erdelyi, A. Higher Transcendental Functions, Vol. I, (1954).
3. Saxena, R. K. *Proc. Nat. Inst. Sc., India,* Vol. 28-A, No 4, (1960).

ON INTEGRATION WITH RESPECT TO PARAMETERS

By

K. C. GUPTA

Lecturer in Mathematics, M. R. Engineering College, Jaipur

[Received on 23rd April, 1965]

ABSTRACT

The aim of this note is to integrate the H-function defined by FOX [4, p. 408] with respect to parameters. The method used here is that of operational calculus. Since the H-function is a generalization of Meijer's G-function [1, p. 207], integration with respect to parameters and the Mellin transform of many well-known functions follow as particular cases of our result.

1. The following notations will be used in this paper

$$(1{\cdot}1) \qquad L\,\{\, f(x)\,;\, p\,\} = \int_0^\infty e^{-px} f(x)\, dx,$$

$$(1{\cdot}2) \qquad K\,\{\, f(x)\,;\, \nu\,;\, p\,\} = \int_0^\infty (px)^{\frac{1}{2}}\, k_\nu(px)\, f(x)\, dx.$$

Formula (1·2) is a generalization of (1·1) as given by Meijer,[6] we have

$$(1{\cdot}3) \qquad \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K\,\{\, f(x)\,;\, \pm\tfrac{1}{2}\,;\, p\,\} = L\,\{\, f(x)\,;\, p\,\}.$$

2. Definition and properties of H-function :

The definition of the H-function used here has slight difference in the parameters to that given by FOX. We define

$$(2{\cdot}1) \qquad H_{s,\,q}^{m,n}\left[\, x \,\middle|\, \begin{matrix} (a_1, e_1), (a_2, e_2), \ldots, (a_s, e_s) \\ (b_1, f_1), (b_2, f_2), \ldots, (b_q, f_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j\xi) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j\xi) \prod_{j=n+1}^{s} \Gamma(a_j - e_j\xi)}\, x^{\xi}\, d\xi,$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$; $0 \leqslant n \leqslant s$; e's and f's are all positive ; L is a suitable contour such that the poles of $\Gamma(b_j - f_j\,\xi), j = 1, 2, \ldots m$, lie on one side of the contour and those of $\Gamma(1 - a_j + f_j\,\xi), j = 1, 2, \ldots n$, lie on the other side. Also the parameters are so restricted that the integral on the right hand side of (2·1) is convergent.

If we apply the formula :

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} \; m^{mz-\frac{1}{2}} \prod_{j=0}^{m-1} \left(z + \frac{j}{m}\right)$$

to the right hand side of (2·1) we easily arrive at the following simple but interesting formula connecting the H–function and the G–function.

(2·2) $$H^{m,\,n}_{s,\,q}\left[x \,\middle|\, \begin{matrix}(a_1, l), \ldots, (a_s, l)\\(b_1, l), \ldots, (b_q, l)\end{matrix}\right] = \frac{1}{l}\, G^{m,\,n}_{s,\,q}\left[x^{\frac{1}{l}} \,\middle|\, \begin{matrix}a_1, \ldots, a_s\\ b_1, \ldots, b_q\end{matrix}\right],$$

where l is a positive integer.

3. The following results proved recently by the author[5] will be required later on

(3·1) $$K\left\{ x^l\, H^{m,\,n}_{s,\,q}\left[zx^\sigma \,\middle|\, \begin{matrix}(a_1, e_1), \ldots (a_s, e_s)\\(b_1, f_1), \ldots (b_q, f_q)\end{matrix}\right] ; \nu ; p \right\}$$

$$= \frac{2^{l-\frac{1}{2}}}{p^{l+1}}\; H^{m,\,n+2}_{s+2,\,q}\left[z\left(\frac{2}{p}\right)^{\sigma} \,\middle|\, \begin{matrix}\left(\frac{1}{4}-\frac{l}{2}-\frac{\nu}{2}, \frac{\sigma}{2}\right), \left(\frac{1}{4}-\frac{l}{2}+\frac{\nu}{2}, \frac{\sigma}{2}\right), (a_1, e_1), \ldots, (a_s, e_s)\\ (b_1, f_1), \ldots, (b_q, f_q)\end{matrix}\right],$$

provided that $\sigma > 0$, $R(p) > 0$, $R(l + \frac{3}{2} \pm \nu) > 0$ and one of the following sets of conditions are satisfied :

(*i*) $\lambda > 0$, $|\arg(z)| < \frac{1}{2}\lambda\pi$,

(*ii*) $\lambda \geqslant 0$, $|\arg(z)| \leqslant \frac{1}{2}\lambda\pi$, $R(\mu+1) < 0$ and $R(\mu + l + \frac{3}{2}) < 0$.

In (3·1) λ and μ stand for the quantities

$$\sum_{j=1}^{n}(e_j) - \sum_{j=n+1}^{s}(e_j) + \sum_{j=1}^{m}(f_j) - \sum_{j=m+1}^{q}(f_j)$$

and

$$\tfrac{1}{2}(s-q) + \sum_{j=1}^{q}(b_j) - \sum_{j=1}^{s}(a_j)$$

respectively.

(3·1) reduces to the following result if we take $\nu = \pm\frac{1}{2}$ in it.

(3·2) $$L\left\{ x^l\, H^{m,n}_{s,\,q}\left[z\,x^\sigma \,\middle|\, \begin{matrix}(a_1, e_1), \ldots, (a_s, e_s)\\(b_1, f_1), \ldots, (b_q, f_q)\end{matrix}\right] ; p \right\}$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2^{l-\frac{1}{2}}}{p^{l+1}}\; H^{m,n+2}_{s+2,\,q}\left[z\left(\frac{2}{p}\right)^{\sigma} \,\middle|\, \begin{matrix}\left(-\frac{l}{2}, \frac{\sigma}{2}\right), \left(\frac{1}{2}-\frac{l}{2}, \frac{\sigma}{2}\right), (a_1, e_1), \ldots, (a_s, e_s)\\ (b_1, f_1), \ldots, (b_q, f_q)\end{matrix}\right]$$

where $\sigma > 0$, $R(p) > 0$ and $R(l+1) > 0$, the other conditions of validity being one of the conditions of (3·1)

4. Theorem. If $f(x) = 0(x^{l})$ for small x and $f(x) = 0(e^{-\alpha x} x^{m})$ for large x, $R(l+\frac{3}{2}) > | R(\nu) |$, $R(p+\alpha) > 0$, $R\left(\alpha + \frac{t^2+p^2}{2t}\right) > 0$ and $| \arg(p) | < \frac{\pi}{4}$, then

(4·1)
$$\int_0^\infty x^{\frac{1}{2}} e^{\frac{-x}{2t}(t^2+p^2)} f(x)\ d(x)$$

$$= 2p^{-\frac{1}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{t}{p}\right)^{\nu} K\{ f(x) ; \nu ; p \}\ d\nu.$$

Proof: We have [2, p. 82]

$$K_\nu(px) = \tfrac{1}{2}p^\nu \int_0^\infty e^{-\frac{1}{2}x\left(t+\frac{p^2}{t}\right)} t^{-\nu-1}\, dt,$$

where $R(x) > 0$, $R(p) > 0$.

Therefore

(4·2)
$$\int_0^\infty (px)^{\frac{1}{2}} k_\nu(px) f(x)\, dx$$

$$= \int_0^\infty (px)^{\frac{1}{2}} f(x) \left[\tfrac{1}{2}p^\nu \int_0^\infty e^{-\frac{1}{2}x\left(t+\frac{p^2}{t}\right)} t^{-\nu-1}\, dt \right] dx.$$

On inverting the order of integration in (4·2) we get

(4·3)
$$K\{f(x) ; \nu ; p\}$$

$$= \tfrac{1}{2} p^{\nu+\frac{1}{2}} \int_0^\infty t^{-\nu-1} \left[\int_0^\infty e^{\frac{-x}{2t}(t^2+p^2)} x^{\frac{1}{2}} f(x)\, dx \right] dt.$$

Applying inversion formula for the Mellin transform in (4·3) we easily arrive at the said theorem.

Since in (4·2) t integral is absolutely convergent when $| \arg p | < \frac{\pi}{4}$, x–integral on the R. H. S. of (4·2) is absolutely covergent when $R\left(\alpha + \frac{t^2+p^2}{2t}\right) > 0$, $R(l+\frac{3}{2}) > 0$ and the resulting integral on the L. H. S. of (4·2) is convergent when $R(\alpha+p) > 0$, $R(l \pm \nu + \frac{3}{2}) > 0$, the inversion of the order of integration is justified there under the conditions stated with the theorem.

(4·1) was given earlier by Rathie in a different form [7, p. 173].

Example

From (3·1) we have

$$K\left\{x^{l}\ \mathrm{H}_{s,q}^{m,n}\left[z^{-1}x^{\sigma}\left|\begin{matrix}(1-a_1,e_1),\ldots,(1-a_s,e_s)\\(1-b_1,f_1),\ldots,(1-b_q,f_q)\end{matrix}\right.\right];\nu;p\right\}$$

$$=\frac{2^{l-\frac{1}{2}}}{p^{l+1}}\mathrm{H}_{q,s+2}^{n+2,m}\left[z\left(\frac{p}{2}\right)^{\sigma}\left|\begin{matrix}(b_1,f_1),\ldots,(b_q,f_q)\\\left(\frac{3}{4}+\frac{l}{2}+\frac{\nu}{2},\frac{\sigma}{2}\right),\left(\frac{3}{4}+\frac{l}{2}-\frac{\nu}{2},\frac{\sigma}{2}\right),\\(a_1,e_1),\ldots,(a_s,e_s)\end{matrix}\right.\right].$$

Also (3·2) gives

$$\int_0^{\infty}x^{l+\frac{1}{2}}e^{\frac{-x}{2t}(t^2+p^2)}\mathrm{H}_{s,q}^{m,n}\left[z^{-1}x^{\sigma}\left|\begin{matrix}(1-a_1,e_1),\ldots,(1-a_s,e_s)\\(1-b_1,f_1),\ldots,(1-b_q,f_q)\end{matrix}\right.\right]dx$$

$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}2^{l}\left(\frac{2t}{t^2+p^2}\right)^{l+\frac{3}{2}}H_{q,s+2}^{n+2,m}\left[z\left(\frac{t^2+p^2}{4t}\right)^{\sigma}\left|\begin{matrix}(b_1,f_1),\ldots,(b_q,f_q)\\\left(\frac{3}{4}+\frac{l}{2},\frac{\sigma}{2}\right),\left(\frac{5}{4}+\frac{l}{2},\frac{\sigma}{2}\right),\\(a_1,e_1),\ldots,(a_s,e_s)\end{matrix}\right.\right]$$

Using these values in (4·1) we get

(4·4) $$\frac{1}{2\pi i}\int_{c-\infty}^{c+i\infty}\left(\frac{t}{p}\right)^{\nu}\mathrm{H}_{q,s+2}^{n+2,m}\left[z\left(\frac{p}{2}\right)^{\sigma}\left|\begin{matrix}(b_1,f_1),\ldots,(f_q,f_q)\\\left(\frac{3}{4}+\frac{l}{2}+\frac{\nu}{2},\frac{\sigma}{2}\right),\left(\frac{3}{4}+\frac{l}{2}-\frac{\nu}{2},\frac{\sigma}{2}\right),\\(a_1,e_1),\ldots,(a_s,e_s)\end{matrix}\right.\right]d\nu$$

$$=(\pi)^{-\frac{1}{2}}\left(\frac{2pt}{t^2+p^2}\right)^{l+\frac{3}{2}}\mathrm{H}_{q,s+2}^{n+2,m}\left[z\left(\frac{t^2+p^2}{4t}\right)^{\sigma}\left|\begin{matrix}(b_1,f_1),\ldots,(b_q,f_q)\\\left(\frac{3}{4}+\frac{l}{2},\frac{\sigma}{2}\right),\left(\frac{5}{4}+\frac{l}{2},\frac{\sigma}{2}\right),\\(a_1,e_1),\ldots,(a_s,e_s)\end{matrix}\right.\right],$$

provided that one of the following sets of conditions are satisfied:

(*i*) $\lambda>0,\ |\arg(z)|<\frac{1}{2}(\lambda)\pi$

and $|\arg\left(\frac{t}{p}\right)|<\frac{\pi}{2}$;

(*ii*) $(\lambda)\geqslant 0,\ |\arg(z)\leqslant\frac{1}{2}(\lambda)\pi,$

$|\arg\left(\frac{t}{p}\right)|<\frac{\pi}{2}$

and $R(l+\frac{3}{2}+\sum_{j=1}^{s}(a_j)-\sum_{j=1}^{q}(b_j)+\frac{1}{2}q-\frac{1}{2}s)<0,$

where λ stands for the same quantities as in (3·1).

Particular cases :

If we take all e's and f's equal to unity in (4·4) and further put $\sigma = 2$, we get

$$(4\cdot5)\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{p}\right)^{\nu} G^{n+2,\,m}_{q,\,s+2}\left[\frac{4p^2}{a^2}\,\middle|\,\begin{matrix} b_1, b_2, \ldots, b_q \\ \frac{3}{4}+\frac{l}{2}+\frac{\nu}{2},\ \frac{3}{4}+\frac{l}{2}-\frac{\nu}{2},\ a_1, a_2, \ldots, a_s\end{matrix}\right] d\nu$$

$$= (\pi)^{-\frac{1}{2}}\left(\frac{2pt}{t^2+p^2}\right)^{l+\frac{3}{2}} G^{n+2,\,m}_{q,\,s+2}\left[\left(\frac{t^2+p^2}{at}\right)\,\middle|\,\begin{matrix} b_1, b_2, \ldots, b_q \\ \frac{3}{4}+\frac{l}{2},\ \frac{5}{4}+\frac{l}{2},\ a_1, a_2, \ldots, a_s\end{matrix}\right],$$

where $\left|\arg\left(\frac{t}{p}\right)\right| < \frac{\pi}{2}$ and one of the following sets of conditions are satisfied :

(*i*) $m+n-\frac{1}{2}(s+q) > 0$, $\left|\arg\left(\frac{p}{a}\right)\right| < (m+n-\frac{1}{2}s-\frac{1}{2}q)\frac{\pi}{2}$

(*ii*) $m+n-\frac{1}{2}(s+q) \geqslant 0$, $\left|\arg\left(\frac{p}{a}\right)\right| \leqslant (m+n-\frac{1}{2}s-\frac{1}{2}q)\frac{\pi}{2}$ and

$$R\left(l+\frac{3}{2}+\sum_{j=1}^{s}(a_j)-\sum_{j=1}^{q}(b_j)+\frac{1}{2}q-\frac{1}{2}s\right) < 0$$

On specializing the parameters in (4·5) we get the following results ; result given by (4·7) is known [**3**, p. 361].

$$(4\cdot6)\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{p}\right)^{\nu}\Gamma\left(\frac{3}{4}+\frac{l}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{3}{4}+\frac{l}{2}-\frac{\nu}{2}\right){}_2F_1\left(\frac{3}{4}+\frac{l}{2}\pm\frac{\nu}{2};b;\frac{-a^2}{4p^2}\right)$$

$$\times\, d\nu = 2\,\Gamma(l+\tfrac{3}{2})\left(\frac{pt}{p^2+t^2}\right)^{l+\frac{3}{2}}{}_2F_1\left(\frac{3}{4}+\frac{l}{2},\ \frac{5}{4}+\frac{l}{2};b;\frac{-a^2t^2}{(t^2+p^2)^2}\right),$$

where $\left|\arg\left(\frac{t}{p}\right)\right| < \frac{\pi}{2}$, $\left|\arg\left(\frac{p}{a}\right)\right| < \frac{\pi}{2}$.

$$(4\cdot7)\qquad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{p}\right)^{\nu}K_{\nu}\left(\frac{4p}{a}\right)d\nu$$

$$= \tfrac{1}{2}\,e^{\frac{-2p}{a}\left(\frac{t}{p}+\frac{p}{t}\right)},$$

where $\left|\arg\left(\frac{t}{p}\right)\right| < \frac{\pi}{2}$ and $\left|\arg\left(\frac{p}{a}\right)\right| < \frac{\pi}{2}$.

On interpreting (4·4) with the help of the Mellin Transform [2, p. 307], we get

$$(4\cdot 8)\quad \int_0^\infty x^{\nu-1}\left(\frac{x}{x^2+1}\right)^{l+3/2} H^{n+2,\,m}_{q,\,s+2}\left[z\,p^{\sigma}\left(\frac{x^2+1}{4x}\right)^{\sigma} \middle| \begin{matrix} (b_1, f_1), \ldots, (b_q, f_q) \\ \left(\frac{3}{4}+\frac{l}{2}, \frac{\sigma}{2}\right), \left(\frac{5}{4}+\frac{l}{2}, \frac{\sigma}{2}\right), (a_1, e_1), \ldots, (a_s, e_s) \end{matrix} \right] dx$$

$$= \frac{\sqrt{\pi}}{2^{l+\frac{3}{2}}} H^{n+2,\,m}_{q,\,s+2}\left[z\left(\frac{p}{2}\right)^{\sigma} \middle| \begin{matrix} (b_1, f_1), \ldots, (b_q, f_q) \\ \left(\frac{3}{4}+\frac{l}{2}+\frac{\nu}{2}, \frac{\sigma}{2}\right), \left(\frac{3}{4}+\frac{l}{2}-\frac{\nu}{2}, \frac{\sigma}{2}\right), (a_1, e_1), \ldots, (a_s, e_s) \end{matrix} \right],$$

the conditions of validity being those of (4·4).

If we take all e's and f's equal to 1 and $\sigma = 2$ in (4·8) we get the Mellin transform of the G–function, in which the argument of the G–function contains a factor $\left(\frac{x^2+1}{x}\right)^{\sigma}$. The Mellin transform of the G–functions thus obtained reduces to a known result [7, p. 179] on specializing the parameters in it.

ACKNOWLEDGEMENT

The author is indebted to Dr. K. C. Sharma for his kind help during the preparation of this note. Thanks are also due to Principal V. G. Garde and Prof. Rathie for their encouragement.

REFERENCES

1. Bateman Manuscript Project, Higher Transcendental Functions, Vol. 1 McGraw Hill, New York (1953).
2. Bateman Manuscript Project, Higher Transcendental Functions Vol. 2, McGraw Hill, New York (1953).
3. Bateman Manuscript Project, Tables of Integral Transforms Vol. I, McGraw Hill, New York (1954).
4. Fox, C. The G and H functions as symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.*, 98 : 408 (1961).
5. Gupta. K. C. On the H–function. *Annales de la Societe Scientifique de Bruxelles*, (In press).
6. Meijer, C. S. *Proc. Nederl. Akad. Wetensch*, 44 : 727 – 737, (1941).
7. Rathie, C. B. *Proc. Glasg. Math. Assn.*, Pt. IV, 2 : 173, (1956).

CHEMICAL EXAMINATION OF THE SEEDS OF HOLOPTELIA INTEGRIFOLIA PLANCH—STUDY OF FAT

By

S. N. KHANNA, P. C. GUPTA *and* J. D. TEWARI

Chemical Laboratories, University of Allahabad, Allahabad

[Received on 20th November, 1965]

ABSTRACT

The fatty acid composition of the oil from *Holoptelia integrifolia* has been determined. The fatty acids after segregating into solid and liquid fractions have been further fractionated into their respective components by the urea adduct formation method. The fatty acids components have been found Palmitic, 34·37% stearic, 22·12% Oleic, 37·98% and linoleic, 1·82%. The unsaponifiable matter was found to be β-sitosterol.

Holoptelia integrifolia Planch[1] (H. chilbil, N. O. urticaceae) is a large spreading globrous deciduous tree (15–18 meters high) and it is distributed throughout India. The tree bark is squeezed out and applied to rheumatic swellings, exhausted bark is then powdered and applied over the part covered by the sticky juice. The seeds are edible and tasty.

Preliminary chemical examination showed the presence of fat while alkaloid, glycoside and tannins were found to be absent. The seeds yield a high percentage of fat.

EXPERIMENTAL

1/2 Kg. of the powdered seeds were extracted with petroleum ether (40–60°). After distilling off the solvent almost a white fatty solid mass (174 gm., 34·8%) was obtained. The physical and chemical constants of the fat were found to be m. p., 29° ; acid value, 9·45 ; sap. value, 195·2 ; I–value (Hanus), 42·2 ; acetyl value, 2·3 ; unsaponifiable matter, 0·8%.

135 gm. of the fat were saponified and the fatty acids were recovered by the usual method. The mixed fatty acids (130 gm. ; S. V., 207·4 ; I. V., 52·2) ; were segregated into solid acids (57·0 gm. ; 44·19% ; S. V. 212·2 ; I. V. 2·82) and liquid acids (72·1 gms ; 55·81% ; S. V., 202·4 ; I. V., 68·21) components by Twitchell Lead Salt alcohol process[2] modified by Hilditch and co-workers.[3]

The solid and liquid acids were further fractionated with urea as following :

(*a*) *Fractionation of solid acids :*

A series of 250 ml. ground glass stoppered conical flasks numbered S_1 to S_{10} were used, flask S_1–S_9 for adduct formation while S_{10} for raffinate fractions. Flasks S_2–S_9 were charged with 6 gm. of urea each. A saturated solution of urea in methanol : ethyl acetate (7 : 3) was prepared at room temperature and solid acids (28·0 gm, S. V., 212·2 ; I. V. 2·82) were dissolved in 150 ml. of this solution in flask S_1 and 6·0 gm. of urea was added to it. After a little warming, it was kept in cold for three hours when adduct formation was observed. The

supernatent liquid was transferred to flask S_2. 150 ml. of the saturated solution of urea were again added to flask S_1 and adduct formation was carried out in flask S_1 and S_2. The supernatent liquid from the flask S_2 was then transferred to S_3 and from the S_1 to S_2. 150 ml. of saturated urea solution was again added to flask S_1 and adduct formation was carried in S_1, S_2, S_3 and S_4. This process was continued upto flask S_9. The supernatent liquid each time from S_9 was collected in S_{10} as raffinates. The solvent from S_{10} was distilled and fatty acids recovered from each flask by treating with warm acidulated water followed by extraction with ether. The results are presented in table 1.

TABLE 1

Fractionation of Solid acids with urea

(Acid taken = 28·0 gm., I. V. = 2·82, Sap. V. = 212·2)

Adducts	Wt. of fraction	S. V.	M.Wt.	I. V.	Palmitic	Stearic	Oleic
S_1	9·86 gm.	209·3	267·4	0·9	5·47 gm.	4·29 gm.	0·10 gm.
S_2	3·70 gm.	209·3	267·4	0·8	2·06 gm.	1·61 gm.	0·03 gm.
S_3	2·80 gm.	216·0	259·2	1·4	2·40 gm.	0·36 gm.	0·04 gm.
S_4	2·10 gm.	221·0	256·4	3·1	2·03 gm.	–	0·07 gm.
S_5	1·85 gm.	212·0	264·1	1·00	1·25 gm.	0·58 gm.	0·02 gm.
S_6	1·35 gm.	199·8	280·0	2·4	1·15 gm.	0·16 gm.	0·04 gm.
S_7	1·60 gm.	217·0	258·0	2·0	1·43 gm.	0·13 gm.	0·04 gm.
S_8	1·45 gm.	218·2	256·6	1·6	1·38 gm.	0·4 gm.	0·03 gm.
S_9	1·45 gm.	222·3	254·9	1·9	1·42 gm.	–	0·03 gm.
Raffinate							
S_{10}	1·30 gm.	201·2	278·3	30·4	0·16 gm.	0·70 gm.	0·44 gm.
	27·46 gm.				18·75 gm.	7·87 gm.	0·84 gm
	% acid				68·28 gm.	28·66 gm.	3·06 gm.
	% in mixed acids				30·17 gm.	12·67 gm.	1·35 gm.

A qualitative separation of solid acids was done by paper-chromatography (Whatman No. 1 filter paper impregnated in 10% solution of liquid paraffin in benzene ; glacial acetic acid as solvent ; 5% acqueous copper acetate, followed by 6% aqueous potassium ferrocyanide as developer[4,5,6]) where it showed only two distinct spots corresponding their Rf to Palmitic and stearic acids.

(*b*) *Fractionation of liquid acids* :

Liquid acids (52·0 gm ; S. V., 202·4 ; I. V. 68·21) were fractionated into nine adducts and nine raffinates fractions as in the case of solid acids except raffinates were collected separately, using 6 gm of urea and 150 ml of saturated solution of urea in methanol ; ethyl acetate (7 : 3) at each stage. The results have been presented in table 2.

TABLE 2

Fractionation of liquid acids with urea.

(Acid taken = 520 gm. ; S. V. = 202·4 ; O. V. = 68·21)

Adducts	Wt. of fraction	S. V.	M.Wt.	I. V.	Palmitic	Stearic	Oleic	Linoleic
L_1	6·9 gm.	205·0	273·1	55·3	1·01 gm.	1·79 gm.	3 97 gm.	0·13 gm.
L_2	6·2 gm.	195·4	286·5	60·2	–	1·99 gm.	4·21 gm.	–
L_3	3·4 gm.	200·4	279·4	61·5	0·16 gm.	0·94 gm.	2·30 gm.	–
L_4	3·6 gm.	201·2	278·3	65·6	0·19 gm	0·82 gm.	2·56 gm.	0·03 gm.
L_5	3·6 gm.	203·6	275·4	72·3	0·23 gm.	0·55 gm.	2·75 gm.	0·07 gm.
L_6	3·3 gm.	206·2	271·5	76·2	0·25 gm.	0·36 gm.	2·59 gm.	0·10 gm
L_7	3·2 gm.	203·4	275·2	75·0	0·17 gm.	0·43 gm.	2·54 gm.	0·06 gm.
L_8	3·3 gm.	201·0	278·6	75·2	0·10 gm.	0·48 gm.	2·69 gm.	0·03 gm.
L_9	2·5 gm.	201·8	277·5	75·8	0 09 gm.	0 34 gm.	2·04 gm.	0·03 gm.
Raffinate								
L_{18}	6·5 gm.	210·0	266·6	62·6	1·84 gm.	1·27 gm.	2·28 gm.	1·11 gm.
L_{17}	5·9 gm.	203·0	275·8	70·7	1·00 gm.	0·37 gm.	4·43 gm.	0 10 gm.
L_{16}	1·5 gm.	195·4	284·5	86·1	–	0·07 gm.	1·43 gm.	–
L_{15}	1·0 gm.	195·0	385·1	95·0	–	–	0·95 gm.	0·05 gm.
L_{14}	0·1 gm.	–	–	–	–	–	–	–
L_{13}	0·05 gm.	–	–	–	–	–	–	–
L_{12}	–	–	–	–	–	–	–	–
L_{11}	–	–	–	–	–	–	–	–
L_{10}	–	–	–	–	–	–	–	–
Total Wt. 50·9 gm.					50·04 gm.	94 1 gm.	34·74 gm.	1·71 gm.
% acid					9·90 gm.	18·47 gm.	68·24 gm.	3·39 gm.
% in mixed acid					5·52 gm.	10·30 gm.	38·10 gm.	1·89 gm.

Examination of unsaponifiable matter :

After saponification of the oil unsaponifiable matter was removed by shaking the aqueous solution of soap with ether and then with ethyl acetate. The ether soluble fraction was chromatographed over neutral alumina. Benzene and Petroleum ether mixture (2 : 9) furnished a crystalline product and no other fraction was recovered. It was recrystallised from methanol, m. p. 134–136°C. The compound responded to Burchard–Liebermann reaction for sterols. An acetyl derivative m. p 126°C was obtained and it was inferred to be β–stitosterol, which was further confirmed by m. m. p. with an authentic sample. The ethyl acetate soluble fraction could not be worked out owing to its very minute amount.

ACKNOWLEDGMENT

Thanks are due to Dr. K. C. Srivastava, Department of Chemistry, Allahabad University for his active help. One of the authors (S. N. K.) expresses his gratitude to the C. S. I. R. for the grant of J. R. F.

REFERENCES

1. Kirtikar, K. R. and Basu. B. D *Indian medicinal plants*, Vol. III, 2293.
2. Twitchell, E · J. *Ind. Eng. Chem.*, 13 : 806. (1921).
3. Hilditch. The chemical constitution of Natural Fats, Chapman and Hall, 3rd Edition, 577, (1956).
4. Inouye, X. and Noda, M. *J. Agr. Chem. Soc., Japan*, 26 : 634. (1952).
5. *Ibid* ; 27 : 50, (1953).
6. Spiteri, J. *Bull. Soc. Chim. biol.*, 36 : 1355, (1954).

ON AN INTEGRAL TRANSFORM

By

S. K. KULSHRESHTHA

Department of Mathematics, Rajasthan University, Jaipur

[Received on 28th September, 1965]

ABSTRACT

In this paper, the generalization of well known Laplace Transform

$$\psi(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt$$

in the form

$$\phi(p) = p\int_0^\infty (2pt)^{-\frac{1}{4}}\; M_{k,m}(2pt)\, h(t)\, dt$$

has been studied. In Section I some elementary properties of the Kernel involved in the generalized Transform and some elementary theorems on the same Transform are given. In Section II two theorems relating Laplace Transform and generalized transform are obtained. One infinite integral involving Bessel's function is also evaluated with the help of the given theorem.

INTRODUCTION

The classical Laplace Transform

$$L\,[h(t)\,;p] = p\int_0^\infty e^{-pt}\, h(t)\, dt \qquad \ldots\ldots \quad (1\cdot1)$$

has been generalised by various authors. In this paper we shall study the generalisation of this Transform in the form

$$\phi\,(p\,;\,k,m) = M\,[\,h(t)\,;\,k,m] = p\int_0^\infty (2pt)^{-\frac{1}{4}}\, M_{k,m}(2pt)\, h(t)\, dt \qquad \ldots\ldots \quad (1\cdot2)$$

which reduces to (1·1) when $k = -m = \frac{1}{4}$ due to the identity [2, p. 432]

$$M_{\frac{1}{4},\,-\frac{1}{4}}(x) \equiv x^{\frac{1}{4}}\;\; e^{-x/2}. \qquad \ldots\ldots \quad (1\cdot3)$$

This Transform can also be related to Whittaker Transform given by Varma[6] in the form

$$W\,[\,h(t)\,;k,m] = p\int_0^\infty (2pt)^{-\frac{1}{4}}\, W_{k,m}(2pt)\, h(t)\, dt \qquad \ldots\ldots \quad (1\cdot4)$$

due to the identity (2, p. 430)

$$W_{k,m}(x) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\, M_{k,m}(x) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\, M_{k,-m}(x). \qquad \ldots\ldots \quad (1\cdot5)$$

The object of this paper is to study the Transform (1·2). In the Section I some elementary Lemmas are given and in Section II there are Theorems relating this said Transform with Laplace Transform and with the help of these theorems some infinite integrals are also evaluated.

Section I

2. Following are some known elementary properties of the kernel involved in the Transform (1·2) which will be used latter on :

(*a*) The result (5, p. 24)

$$\frac{d}{dx}\left[x^{-\frac{1}{4}} M_{k,m}\ (x)\right] = x^{-5/4}\left[\left(\tfrac{1}{2}+k+m\right) M_{k+1,m}\ (x)\right.$$

$$\left. -\left(k+\tfrac{1}{4}-x/_{2}\right) M_{k,\ m}\ (x)\right] \qquad \ldots . \quad (2{\cdot}1)$$

(*b*) a particular case of the result (2, p. 419) as

$$\int_0^\infty x^{-\frac{1}{4}} M_{k,m}\ (x)\ dx = (-2)^{m+5/4}\ \Gamma\left(m+\tfrac{5}{4}\right) {}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ m+\frac{5}{4} \\ 2m+1\end{matrix};2\right] \qquad \ldots . \quad (2{\cdot}2)$$

and also the similar results :

(*c*) (1)

$$\frac{d}{2pdt}\left[2pt\right)^{-\frac{1}{4}} M_{k,m}\ (2pt)\left.\right] = (2pt)^{-5/4}\left[\left(\tfrac{1}{2}+k+m\right) M_{k+1,m}\ (2pt)\right. -$$

$$\left. -\left(k+\tfrac{1}{4}-pt\right)\ M_{k,m}\ (2pt)\right], \qquad \ldots . \quad (2{\cdot}3)$$

(2)

$$2t\ \frac{d}{dp}\left[(2pt)^{-\frac{1}{4}} M_{k\ m}\ (2pt)\right] = (2pt)^{-5/4}\left[\left(\tfrac{1}{2}+k+m\right)\ M_{k+1,m}\ (2pt)\right. -$$

$$\left. -\left(k+\tfrac{1}{4}-pt\right)\ M_{k,m}\ \ (2pt)\right] \qquad \ldots . \quad (2{\cdot}4)$$

(*d*)

$$\int_0^\infty\ (2pt)^{-\frac{1}{4}}\ \ M_{k,m}\ \ (2pt)\ dt = \frac{1}{2p}\,(-2)^{m-5/4}\ \ \Gamma\left(m+\tfrac{5}{4}\right)$$

$${}_2F_1\left[\begin{matrix}\frac{1}{2}+k+m,\ m+\frac{5}{4} \\ 2m+1\end{matrix};2\right]. \qquad \ldots . \quad (2{\cdot}5)$$

3. The following are the Lemmas, which are valid under the conditions stated therein. Since they are very elementary we are not giving their proofs but wherever necessary some hints are there. Some of these Lemmas are analogous to that of Laplace Transform.

Lemma 1.

If

$$\phi(p\ ;\ k,\ m) = M[h(t)\ ;\ k,\ m]$$

then

$$\beta\ \phi(p/_{\beta},\ k,\ m) = M[h(\beta t)\ ;\ k,\ m] \qquad \ldots . \quad (3{\cdot}3)$$

Lemma 2.

If

$$\phi\ (p\ ;\ k,\ m) = M\left[\ h(t)\ ;\ k,\ m\right]$$

then

$$M\left[\frac{d}{dt}[t\,h(t)]\,;\,k,m\right] = (k+\tfrac{1}{4})\ \phi(p\,;\,k,m) - (\tfrac{1}{2}+k+m)\,\phi(p\,;\,k+1,m) - -p\,M[t\,h(t)\,;\,k,m] \quad \ldots . \quad (3{\cdot}4)$$

This result can be obtained by integrating by parts the left hand integral considering $\frac{d}{dt}\{t\,h(t)\}$ as second function and then using the result (2·3).

Lemma 3.

If

$$\phi(p\,;\,k,m) = M\,[h\,(t)\,;\,k,m]$$

then

$$p^2\frac{d}{dp}\left[\frac{1}{p}\ \phi(p\,;\,k,m)\right] = (\tfrac{1}{2}+k+m)\,\phi(p\,;\,k+1,m) - (k+\tfrac{1}{4})\,\phi(p\,;\,k,m) + p\,M\,[t\,h\,(t)\,;\,k,m] \quad \ldots . \quad (3{\cdot}5)$$

This result can be obtained by differenciating both sides of the relation (3·1) with respect to p and then using the result (2·4).

Lemma 4.

If

$$W\,[h\,(t)\,;\,k,m] = p\int_0^\infty (2pt)^{-\frac{1}{2}}\,W_{k,m}\,(2pt)\,h(t)\,dt$$

and

$$M\,[h(t)\,;\,k,m] = p\int_0^\infty (2pt)^{-\frac{1}{2}}\,M_{k,m}\,(2pt)\,h(t)\,dt$$

then

$$W\,[h\,(t)\,;\,k,m] = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\,M\,[h(t)\,;\,k,m] + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\,M\,[h\,(t)\,;\,k,-m]. \quad \ldots . \quad (3{\cdot}7)$$

The result can be obtained by using the result (1·5).

Section II

4. In this section we give theorems for the transform (1·2) and two of which are related with Laplace Transform.

Theorem 1.

If

$$\phi_1\,(p\,;\,k,m) = M\,[h_1\,(t)\,;\,k,m] \quad \ldots . \quad (4{\cdot}1)$$

and

$$\phi_2\,(p\,;\,k,m) = M\,[h_2\,(t)\,;\,k,m] \quad \ldots . \quad (4{\cdot}2)$$

then

$$\int_0^\infty \phi_1\,(t\,;\,k,m)\,h_2\,(t)\,\frac{dt}{t} = \int_0^\infty \phi_2\,(t\,;\,k,m),\,h_1\,(t)\,\frac{dt}{t} \quad \ldots . \quad (4{\cdot}3)$$

provided (4·1) and (4·2) exist and the either of the integrals involved in (4·3) is absolutely convergent.

Theorem 2.

If

$$\phi(p\,;\,k,\mu) = M\left[t^{-k-\frac{7}{4}}\,h(1/t)\,;\,k,\mu\right]$$

and

$$\psi(p\,;\,\alpha) = L\left[e^{-\alpha/t}\,h(t)\,;\,p\right], \qquad \ldots. \quad (4{\cdot}4)$$

then

$$\phi(p\,;\,k,\mu) = \frac{(2p)^{5/4}\,\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}-k+\mu)}\int_0^\infty t^{-k-\frac{3}{2}}\,I_{2\mu}(2\sqrt{2pt})\,L\left[e^{-p/x}\,h(x)\,;\,t\right]dt \qquad \ldots. \quad (4{\cdot}5)$$

provided $R(\alpha)>0$, $R(p)>0$, $R(k-\mu)<\frac{1}{2}$, $h(t)$ is independent of α and the integral involved in (4·5) is absolutely convergent.

Proof. We have (1, p. 293)

$$p^{k+1}\,e^{a/p}\,M_{k,\mu}(2\alpha/p) = L\left[\frac{2(2\alpha)^{\frac{1}{4}}\,\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}-k+\mu)}\,t^{-k-\frac{1}{2}}\,I_{2\mu}(2\sqrt{2\alpha t})\,;\,p\right]$$

where $R(\alpha)>0$, $R(p)>0$ and $R(k-\mu)>\frac{1}{2}$. $\qquad \ldots. \quad (4{\cdot}6)$

Applying Parceval's Goldstein theorem (4) to the relations (4·4) and (4·6) we get

$$\int_0^\infty t^k\,M_{k,\mu}(2\alpha/t)\,h(t)\,dt = \frac{2(2\alpha)^{\frac{1}{4}}\,\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}-k+\mu)}\int_0^\infty t^{-k-\frac{3}{2}}\,I_{2\mu}(2\sqrt{2\alpha t})\,\psi(t\,;\,\alpha)\,dt$$

Now replacing t by $(1/t)$ in the left integral, multiplying both sides by $\alpha(2\alpha)^{-\frac{1}{4}}$, replacing α by p and finally on interpreting R. H. S. with the help of (1·2) we obtain the theorem

Corollary 1. In the theorem if we take $k-\mu=\frac{1}{4}$, we get the corollary given by Kulshreshtha.[4]

Corollary 2. Again if we use Lemma 5, we get the corollary given by Kulshreshtha[4] in the form

If

$$\phi(p\,;\,k,\mu) = W\left[t^{-k-\frac{7}{4}}\,h(1/t)\,;\,k,\mu\right]$$

and

$$\psi(p\,;\,\alpha) = L\left[e^{-\alpha/t}\,h(t)\,;\,p\right]$$

then

$$\phi(p\,;\,k,\mu) = \frac{(2p)^{5/4}}{\Gamma(\frac{1}{2}-k\pm\mu)}\int_0^\infty t^{-k-\frac{3}{2}}\,K_{2\mu}(2\sqrt{2pt})\;L\left[e^{-p/x}\,h(x)\,;\,t\right]dt \qquad \ldots. \quad (4{\cdot}7)$$

provided $R(\frac{1}{2}-k\pm\mu)>0$, $R(\alpha)>0$, $R(p)>0$, $h(t)$ is independent of a and the integral involved in (4·7) is absolutely convergent.

Example. If we take

$$h(t) = t^{v-1} e^{-\beta t}$$

then (1, p. 146)

$$e^{-\alpha/t} h(t) = e^{-\alpha/t} \ t^{v-1} \ t^{-\beta t}$$

$$\doteqdot 2p \ \alpha^{v/2} (p+\beta)^{1-v/2} K_v \left(2\sqrt{\alpha(p+\beta)}\right)$$

$$= \psi(p, \alpha)$$

where $R(\alpha) > 0$, $R(k) > 0$ and $R(p) > 0$.

Also we have

$$\phi(p\,;\, k, \mu) = M\left[\, t^{k-\frac{7}{4}} h(1/t)\,;\, k, \mu\,\right]$$

that is

$$\phi(p\,;\, k, \mu) = p\,(2p)^{-\frac{1}{4}} \int_0^\infty t^{-k-v-1} e^{-\beta/t} M_{k,\mu}(2pt)\, dt$$

$$= \frac{p\,(2p)^{-\frac{1}{4}}\,\Gamma(2\mu+1)}{\beta^{k+v+1}\,\Gamma(\frac{1}{2}+k+\mu)} \sum_{r=0}^{\infty} \frac{(\beta p)^r}{\underline{|r}}\, G^{2\,1}_{1\,3}\left(2\beta p \,\middle|\, \begin{matrix} 1-k \\ 2+k+v-r,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right). \quad \ldots\ldots (4{\cdot}8)$$

where $R(p) > 0$ and $R(k-\mu) < \frac{1}{2}$.

Now using the value of $\psi(p\,;\,\alpha)$ in the relation (4·5) and equating this value of $\phi(p\,;\,k,\mu)$ with the value of $\phi(p\,;\,k,\mu)$ obtained in the relation (4·8), we get

$$\int_0^\infty t^{-k-\frac{1}{2}} (t+\beta)^{1-v/2} K_v\left(2\sqrt{p(t+\beta)}\right) I_{2\mu}\left(2\sqrt{2pt}\right) dt$$

$$= \frac{(2p)^{-v/2-\frac{1}{4}}\,\Gamma(\frac{1}{2}-k+\mu)}{2^{2-v/2}\,\beta^{k+v+1}\,\Gamma(\frac{1}{2}+k+\mu)} \sum_{r=0}^{\infty} \frac{(\beta p)^r}{\underline{|r}}\, G^{2\,1}_{1\,3}\left(2\beta p \,\middle|\, \begin{matrix} 1-k \\ 2+k+v-r,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right), \quad \ldots\ldots (4{\cdot}9)$$

where $R(\beta) > 0$, $R(p) > 0$, $R(k-\mu) < \frac{1}{2}$ and the series is convergent.

In the relation (4·9) if we take $k = \mu + \frac{1}{2}$. We get

$$\int_0^\infty t^{-\mu-1} (t+\beta)^{1-v/2} K_v\left(2\sqrt{p(t+\beta)}\right) I_{2\mu}\left(2\sqrt{2pt}\right) dt = \frac{2^{\mu-1}\, p^{\mu+1}}{\Gamma(2\mu+1)\,\beta^{v/2}} K_{v+2}\left(2\sqrt{p\beta}\right) \quad \ldots\ldots (4{\cdot}10)$$

where $R(p) > 0$ and $R(\beta) > 0$.

Theorem 3.

If

$$\phi(p\,;\, k, \mu) = M\left[\, t^{k-\frac{7}{4}} h(1/t)\,;\, k, \mu\,\right]$$

and

$$\psi(p\,;\,\alpha) = L\left[\, e^{\alpha/t} h(t)\,;\, p\,\right] \quad \ldots\ldots (4{\cdot}11)$$

then

$$\phi(p\,;\, k, \mu) = \frac{p(2p)^{\frac{1}{4}}\,\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+k+\mu)} \int_0^\infty t^{-k-\frac{3}{2}} J_{2\mu}\left(2\sqrt{2pt}\right) L\left[\, e^{p/x} h(x)\,;\, t\,\right] dt \quad \ldots (4{\cdot}12)$$

provided $R(p) > 0$, $R(\alpha) > 0$, $R(k+\mu) > -\frac{1}{2}$, $h(t)$ is independent of α and the integral involved in (4·12) is absolutely convergent.

Proof. We have (1, p. 185)

$$\frac{\Gamma(\frac{1}{2}+k+\mu)}{(2\alpha)^{\frac{1}{2}}\,\Gamma(2\mu+1)}\, p^{1-k}\, e^{-\alpha/p}\, M_{k,\,\mu}\,(2\alpha/p) = L\,[\,t^{k-\frac{1}{2}}\, J_{2\mu}\,(2\sqrt{2at})\,;\,p] \qquad \ldots\ldots (4{\cdot}13)$$

where $R(\alpha) > 0$, $R(p) > 0$ and $R(k+\mu) > -\frac{1}{2}$.

Applying Parseval's Goldstein theorem (4) to the relations (4·11) and (4·12) we get

$$\int_0^\infty t^{-k}\ M_{k,\mu}\ \ (2\alpha/t)\ \ h(t)\ \ dt = \frac{(2\alpha)^{\frac{1}{2}}\ \Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+k+\mu)} \int_0^\infty t^{-k-\frac{3}{2}} J_{2\mu}\,(2\sqrt{2at})\ \psi(t,\,\alpha)\ dt.$$

Now replacing t by $1/t$ in the left integral, multiplying both sides by $\alpha(2\alpha)^{-\frac{1}{2}}$ then replacing α by p and finally on interpreting R. H. S. with the help of (1·7) we obtain the theorem.

Corollary.—In the theorem 3 if we take $k = -\mu = \frac{1}{4}$ we get

If

$$\phi(p) = L\,[t^{-3/2}\,h(1_{/t})\,;\,p\,]$$

and

$$\psi(p\,;\,\alpha) = L\,[\,e^{\alpha/t}\ \ h(t)\ ;\,p\,]$$

then

$$\phi(p) = p(2p)^{\frac{1}{2}}\,\pi^{\frac{1}{2}} \int_0^\infty t^{-5/4}\, J_{-\frac{1}{2}}\,(2\sqrt{2pt})\, L\,[\,e^{p/x}\,h\,(x)\,;\,t\,] \qquad \ldots\ldots (4{\cdot}14)$$

provided $R(\alpha) > 0$, $R(p) > 0$, $h(t)$ is independent of α and the integral involved in (4·14) is absolutely convergent.

ACKNOWLEDGEMENT

The author is grateful to Dr. K. C. Sharma for his guidance and keen interest during the preparation of this paper.

REFERENCES

1. Erdelyi, A. "Tables of Integral Transforms" Vol. I. *McGraw Hill, New York*, (1954).
2. Erdelyi, A. "Tables of Integral Transforms" Vol. II. *McGraw Hill, New York*, (1954).
3. Goldstein, S. *Proc. Lond. Soc.* (2), **34** : 103–125.
4. Kulshreshtha, S. K. "Theorems on Whittaker Transform & infinite integrals involving Bessel function", Communicated for publication to *Seminerio Matematico de Barcelona, Spain.*
5. Slater, L. J. "Confluent Hypergeometric Function" *Cambridge University Press, London.*
6. Varma, R. S. *Current Sciences*, **15** : 16- 17, (1947).

AN ELECTROMETRIC STUDY ON THE QUANTITATIVE ESTIMATION OF THORIUM AS MOLYBDATE

By

SARJU PRASAD *and* MISS SHYAMALA DEVI

Chemical Laboratories, Banaras Hindu University, Varanasi–5

[Received on 25th September, 1965]

ABSTRACT

Results of a series of conductometric titrations of thorium nitrate and sodium molybdate are reported. The results of direct titrations lend support to the formation of normal thorium molybdate. Reverse titrations, however, do not give accurate results.

INTRODUCTION

Thorium has been estimated potentiometrically and conductometrically by titrating its acetate solution with alkali hydroxides or the nitrate solution with hydroxides, carbonates, phosphates, arsenate, vanadate, selenate, molybdate and tungstate of sodium and with tellurate, chromate, ferri– and ferrocyanides of potassium[1,2]. Electrometric titrations[3] of the salts of lanthanum, cerium and thorium with solutions of sodium and ammonium oxalates and those of thorium chloride with oxalic[4] and sebacic[5] have been carried out successfully. Kabadi and coworkers[6,7] studied electrometrically the formation of colloidal thorium molybdate at pH 4·13 and the interaction between thorium nitrate and mono and dipotassium arsenates. By conductometric and pH metric titrations[8] of thorium nitrate with $Na_2H_2P_2O_6$ solution the formation of $ThP_2O_6.2H_2O$ has been confirmed. Gaur and Bhattacharya[9] studied the formation of $Th[Fe(CN)_6]$ by the conductometric and potentiometric titrations. Thorium when titrated conductometrically[10] with selenous acid at pH 2·0 gives accurate values. Thorium has been estimated conductometrically as normal metavanadate[11], tellurite[12] and tungstate[13] in this laboratory. The work has now been extended to the micro-determination of thorium as normal thorium molybdate.

EXPERIMENTAL

Sodium molybdate (Riedel—A. R. quality) and thorium nitrate (May and Baker Reagent quality) were dissolved separately in conductivity water and thorium estimated as ThO_2[14] by the oxalate method. Solutions of different concentrations were obtained by appropriate dilutions.

Titrations were carried out at 30 $\pm$ 0·5°C in a Jena Beaker (100 ml) with a dip cell of type PR 9510 having the cell constant 1·44 Philips Conductivity bridge type PT 9500, a 220 volts A. C. line operated direct reading electronic instrument with a magic eye as an indicator was used for the electrolytic resistance measurements. The readings were recorded in ohms at an output frequency of 50 cycles/second.

Effect of Ethanol

When titrations were carried out in purely aqueous medium correct results were not obtained. A number of titrations were therefore carried out at various ethanol concentrations ranging from 5–40% and the results improved (*cf.* Fig. 1).

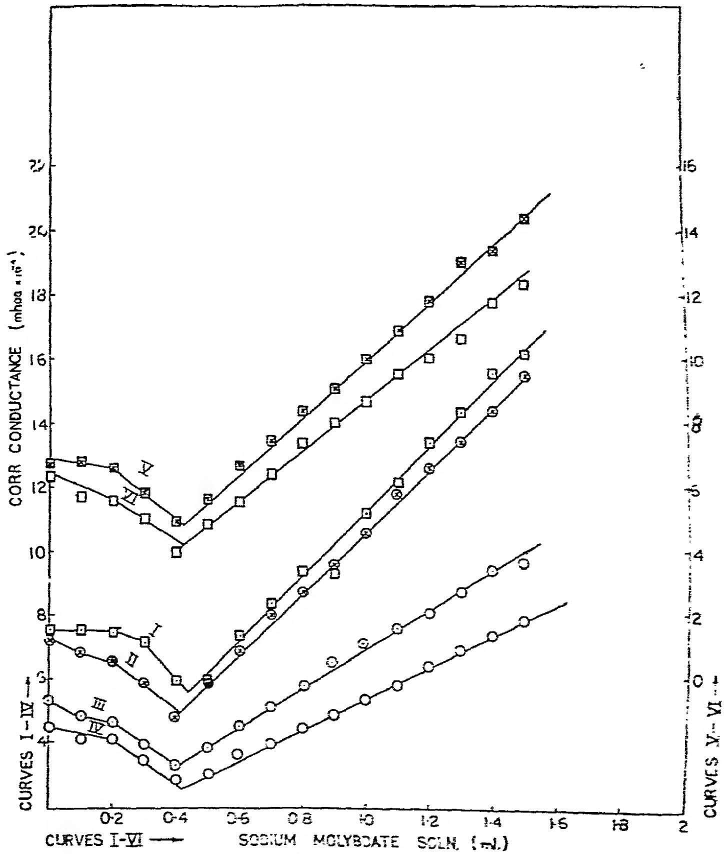


FIG I
EFFECT OF ETHANOL

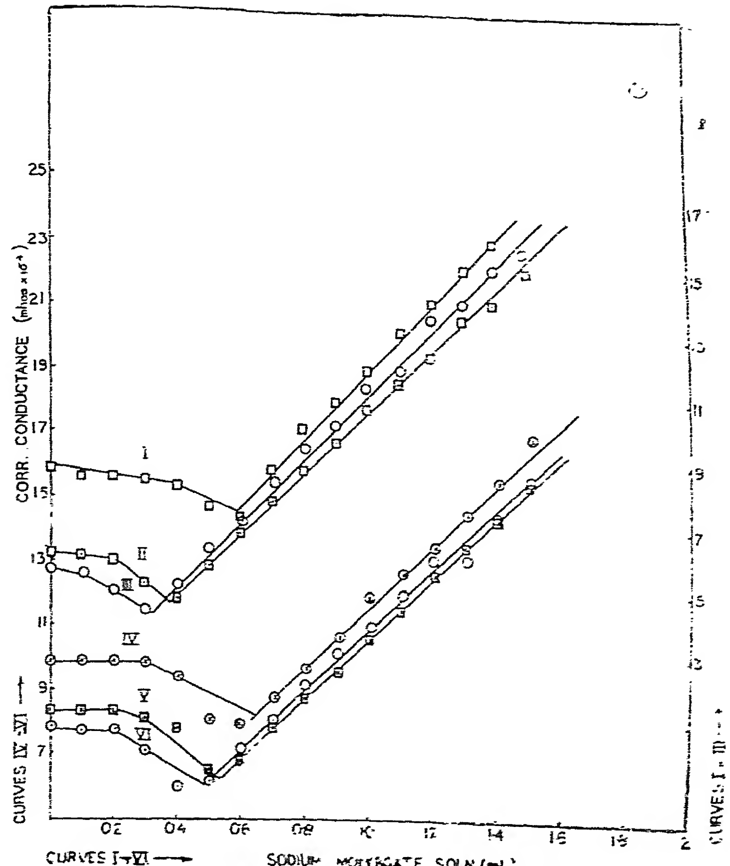


FIG II
DIRECT CONDUCTOMETRIC TITRATION

Direct Conductometric Titrations

A known volume of thorium nitrate solution with an overall 5% EtOH was taken in the beaker, placed on the magnetic stirrer, the dip cell properly dipped in the solution and the initial conductance recorded. Sodium molybdate solution was added from a micro burette in small amounts to the thorium nitrate solution, the mixture was stirred well and the readings recorded after allowing the precipitate to settle, the temperature being maintained at 30± 0·5°C. Conductances were plotted against volumes of the titrant. The results are given in Fig. II Curves I–VI.

Reverse Conductometric Titrations

In reverse titrations sodium molybdate solution with an overall 5% alcohol was taken in the beaker and titrated against thorium nitrate solution. Results are shown in Fig. III Curves I—III.

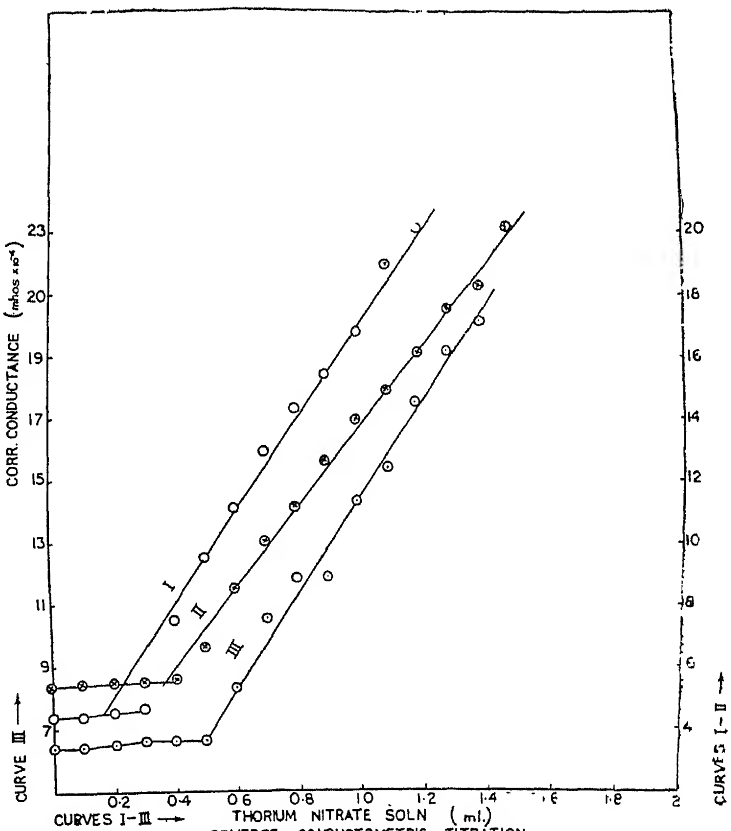


FIG III REVERSE CONDUCTOMETRIC TITRATION

EFFECT OF ETHANOLIC CONCENTRATION

Molarity of the diluted sodium molybdate solution = 0·1991 M
Molarity of diluted thorium nitrate solution = 0·01068 M

S. No.	Alcoholic conc. %	Volume of $Th(NO_3)_4$ soln.	Molarity of $Th(NO_3)_4$ soln.	Vol. of Na_2MoO_4 soln.	Molarity of Na_2MoO_4 soln.	Molar ratio of Th : Mo
1	0	60 ml	0·0007120	0·44	0·1991	1 : 2·050
2	5	,,	,,	0·42	,,	1 : 1·957
3	10	,,	,,	0·42	,,	1 : 1·957
4	20	,,	,,	0·42	,,	1 : 1·957
5	30	,,	,,	0·418	,,	1 : 1·948
6	40	,,	,,	0·42	,,	1 : 1·957

DIRECT CONDUCTOMETRIC TITRATION

S. No.	Vol. of Th. soln.	Molarity of $Th(NO_3)_4$	Vol. of molybdate soln.	Molarity of molybdate soln.	Molar ratio of Th : Mo
1	60 ml	0·0010680	0·64	0·1991	1 : 1·988
2	,,	0·0009790	0·59	,,	1 : 2·000
3	,,	0·0008900	0·535	,,	1 : 1·995
4	,,	0·0008009	0·482	,,	1 : 1·996
5	,,	0·0007120	0·42	,,	1 : 1·957
6	,,	0·0006230	0·375	,,	1 : 1·997
7	,,	0·0005339	0·32	,,	1 : 1·988

REVERSE CONDUCTOMETRIC TITRATIONS

S. No	Vol of $Th(NO_3)_4$ soln.	Molarity of thorium soln.	Vol. of molybdate soln.	Molarity of molybdate soln.	Molar ratio of Th : Mo
1	0·500	0·1068	60 ml	0·001991	1 : 2·238
2	0·365	,,	,,	0·001659	1 : 2·554
3	0·160	,,	,,	0·001327	1 : 4·661

DISCUSSION

The reactions involved in the present titrations can be represented as :

$$Th(NO_3)_4 + 2Na_2MoO_4 \longrightarrow Th(MoO_4)_2 + 4\overset{+}{Na} + 4\overset{-}{NO_3}$$

When titrations were carried out in purely aqueous medium neither the curves were smooth nor the end points sharp which may be due to the slight solubility of the precipitate or its hydrolysis, which is remedied by the addition of ethanol.

In direct titrations breaks obtained correspond to the molar ratio 1 · 2 of thorium and molybdenum and this supports the formation of normal thorium molybdate. In reverse titrations pre-mature end points were obtained and the ratio of thorium : Molybdenum was more than 2 in all the cases, which may be due to either the adsorption of MoO_4^{--} on precipitated $Th(MoO_4)_2$ or the formation of isopoly acid between $Th(MoO_4)_2$ and Na_2MoO_4.

ACKNOWLEDGMENT

Author's sincere thanks are due to the authorities of the Banaras Hindu University for the facilities provided and to the Ministry of Education Government of India for a research training scholarship to one of them (S. D.).

REFERENCES

1. Britton, H. T. S. and Meek, F. H. *J. C. S.*, 183, (1932).
2. Yoshinaga Oka. *Bull. Tokyo Univ. Eng.*, 8 (21) : (1939) ; *Chem. Zentr.*, I, 3150, (1940).
3. Atanasiu, I. A. *Z. anal. Chem.*, **112** : 19, (1938).
4. Menon, P. G. and Singh, D. *Z. anal. Chem.*, **176** : 264, (1960).
5. Singh. D. and Agarwala, V. S. *J. Sci. Res. Banaras Hindu University*, **12** : No. 293, (1961-62).
6. Pawar, A. A. and Kabadi, M. B. *J. Univ. Bombay.* Sec. A. 20, Part III. 1–19, (1951).
7. Pawar, A. A. and Kabadi, M. B. *J. Univ. Bombay*, Sec. A , 21, Part III, 22, (1952).
8. Moeller, T. and Dawson, G. Q. *J. A C. S.*, **75** : 3572, (1953).
9. Gaur, J. N. and Bhattacharya, A. K. *J. I. C. S.*, **31** : 467, (1954).
10. Deshmukh, G S. Anand, V. D. and Srinivasa, Achar P. *J. Sci. Ind. Res.* (India), **20B** · 356, (1961).
11. Sarju Prasad and Suresh Kumar. *Jour. Indian Chem. Soc.*, Vol. **40** : 534 (1963).
12. Sarju Prasad and Suresh Kumar. *Jour. & Proc. Inst. Chem.* Vol. **35** : p 10, (1963).
13. Sarju Prasad and Devi Shyamala. *J I. C. S.*, Vol., **42** : 859, (1965).
14. Vogel, I. "A Text Book of Quantitative Inorganic Analysis", pp. 472, (1955).

CATALYSED ACETONE : IODINE REACTION IN DIFFERENT SOLVENTS*

By

D. K. BANERJEE *and* A. K. BHATTACHARYA

Chemical Laboratories, University of Saugar, Sagar, M. P. (India)

[Received on 22nd November, 1965]

ABSTRACT

Acetone : Iodine reaction is catalysed in presence of acids. In the present work is used different solvents as reaction media and kinetic data shows that the reaction is of first order in each case. K–values are in the order $HCl > HNO_3 > H_2SO_4 > (COOH)_2$ This has been explained on the basis of Intermediate Compound Theory. By using different solvents reaction with water is ruled out as an influencing factor causing removal of iodine.

The reaction between acetone and iodine has been studied for a long time. In a series of publications Dawson and coworkers[1], Bell and Tantram[2] have studied it as an interesting example of catalysis in solution. They found out that in the presence of acids acetone enolises in the first stage as

$$CH_3.\overset{\overset{\displaystyle O}{\|}}{C}-CH_3 \rightleftharpoons CH_2 = \overset{\overset{\displaystyle O-H}{|}}{C}-CH_3$$

which probably occurs as a result of the addition and subsequent removal of proton.

Iodine reacts with the enolic compound instantaneously to form iodo-acetone, and hydriodic acid is liberated.

In the light of the above conclusions, Rao and Bhattacharya[3] concluded that prototropy is practicable in presence of acid catalysts. The present worker has used different solvents as the reaction media.

EXPERIMENTAL

N/50 solutions of iodine were prepared in 5% KI, 10% KI, methyl alcohol, and ethyl alcohol. To 20 ml. of each of these solutions was added 10 ml. of pure acetone mixed with 10 ml. of the respective solvent, 5 ml. of water and 5 ml. of one of the acids N/10 HCl, N/10 HNO_3, N/10 H_2SO_4 and N/10 oxalic acid. The reagents were all of B.D.H. AnalaR quality. The reactants and the reaction mixture were all kept in a thermostat maintained at 20±0·2°C. The rate of reaction was followed at definite time intervals. It was determined by withdrawing 10 ml. of the reaction mixture and titrating the unreacted iodine with standardised sodium thiosulphate solution using starch as indicator. The reaction mixture was chilled everytime before being titrated. The following results were obtained. For brevity, the individual tables of observations are not given but only the results are tabulated, with average K–values, where K is the specific reaction rate calculated by inserting the data in the first-order reaction equation.

---

*Accepted for presentation at the combined 51st and 52nd Session of the Indian Science Congress Association.

TABLE

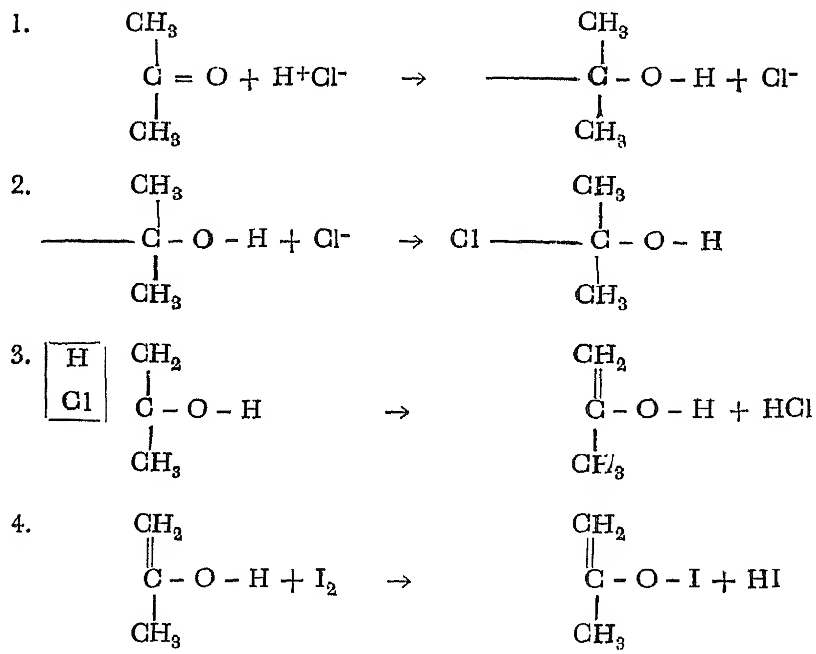
Specific Reaction Rates ($K \times 10^3$)

	Catalysing Acids			
Solvents	N/10 HCl	N/10 HNO_3	N/10 H_2SO_4	N/10 $(COOH)_2$
KI 5%	3·82	3·62	3·15	2·40
KI 10%	4·80	4·60	3·81	1·79
Me OH	7·46	6·36	4·30	2·14
Et OH	4·68	4·62	2·47	1·28

Identical procedure was followed in a blank experiment done in dark. It was noticed that there was no particular difference in the specific reaction rate. Thus the photo-chemical nature of the reaction was ruled out in this case.

DISCUSSION

The data and the results obtained indicate that the reaction is of first order in any solvent in presence of acids. The rate of reaction is proportional to H^+ ions, because the value of K decreases according to the strength of acids. Thus K values are in the order $HCl > HNO_3 > H_2SO_4 >$ Oxalic acid. Dawson has presumed that the over-all velocity of this reaction is due to undissociated acids, but this brings in some anomalies as shown by Narayan Rao and Bhattacharya[4] who gave a scheme of Intermediate Compound theory. Thus in this case,



Similarly, the "Compound formation" with other acids can be explained. As evident from the results given above, in every solvent taken, prototropy takes place soon in case of HCl catalysed reaction because attractive power between H^+ and Cl^- radicals is more than others. This explains the order of K values in case of $HCl > HNO_3 > H_2SO_4 > (COOH)_2$. By using different solvents, side reaction with water is ruled out as an influencing factor causing removal of iodine. Some irregular readings obtained may be explained by the facts that the catalysing action is done by H^+ ions, and these are produced with the dissociation of HI being formed. Thus the concentration of H^+ ions is a function of time and an increase is apparent.

ACKNOWLEDGMENT

One of the authors (D. K. B.) is thankful to the C. S. I. R. for the award of a Senior Research Fellowship.

REFERENCES

1. Dawson, Carter, Dean and Hoskins. *Jour. Chem. Soc*, 1108, 2282, 2872, 3166, (1926).
2. Bell and Tantram *Jour. Chem. Soc*, 370, 374, (1948).
3. Narayan Rao and Bhattacharya. *Jour. Saugar Univ.* 1 : 129, (1952).

COMPLEX DIFFERENTIAL INEQUALITIES AND EXTENSION OF LYAPUNOV'S METHOD

By

S. G. DEO *and* V. LAKSHMIKANTHAM

Department of Mathematics, Marathwada University, Aurangabad

[Received on 29th September, 1965]

ABSTRACT

Extending in a natural way the definitions of stability of solution of ordinary differential inequalities to complex differential inequalities we study in this paper the concept of stability by using the extension of Lyapunov's Method which basically depends on the comparison principle as used in[1] to [5].

1. Introduction

The direct method of Lyapunov and its extension are used as an important tool to study stability problems of differential equations[1,3]. Extending in a natural way the concept of stability of the solutions of complex differential equations and using the extension of Lyapunov's Method, a number of results have been established in[1]. In this paper we generalise the stability notion to complex differential inequalities corresponding to the results in[5]. This enables one to study the stability of a larger class of functions which includes the class of the solutions of complex differential equations and perturbed differential equations. Examples are constructed to illustrate the results.

2. Let D denote the region of the complex plane $|z| \geqslant a$ $(a \neq o)$ and $\alpha \leqslant \arg z \leqslant \beta$, and further let $\bar{C}$ denote the region of the complex y-plane, $|y| < C$, where a, α, β and C are all real numbers. We shall denote $|z|$ by t and $\arg z = \theta$ so that $z = te^{i\theta}$. Suppose that $W_1(t, r)$ is a scalar, real valued function defined and continuous on $I \times R_C^+$ where I is $a \leqslant t < \infty$ and R_C^+ is $o \leqslant r < C$. Let $W_1(t, r)$ be non decreasing in r for each $t \in I$.

We consider the following complex differential inequality

$$(2{\cdot}1) \qquad |y' - f(z, y)| \leqslant W_1(|z|, |y|), \left(' = \frac{d}{dz}\right)$$

where f is regular analytic in $z \in D$ in $y \in \bar{C}$.

A complex function $y(z)$ is said to be a solution of the differential inequality (2·1) if the following conditions are satisfied:

(*I*) $y(z)$ is analytic for all $z \in D$,

(*II*) $y(z_0) = y_0$,

(*III*) $y(z)$ satisfies the differential inequality (2·1) for all $z \in D$.

If $W_1(|z|, |y|) \equiv o$, it is understood that $y(z)$ is a solution of $y' = f(z, y)$.

We wish to study the stability of solutions of the differential inequality (2·1). We shall be interested in proving the following conditions.

(i) For any $\varepsilon > o$ and $z_0 \varepsilon D$, there exists a positive function $\delta = \delta(|z_0|, \varepsilon)$ continuous in $|z_0|$ for each ε such that

$$|y(z)| < \varepsilon, (z \varepsilon D, \ |z| \geqslant |z_0|)$$

whenever

$$|y(z_0)| \leqslant \delta.$$

(ii) For each $\varepsilon > o$, $\alpha > o$ and $z_0 \varepsilon D$ there exists a positive number $T = T(|z_0|, \varepsilon, \alpha)$ such that

$$|y(z)| < \varepsilon, (z \varepsilon D, |z| \geqslant |z_0| + T)$$

whenever

$$|y(z_0)| \leqslant \alpha.$$

(iii) The conditions (i) and (ii) hold simultaneously.

Remark—The definitions given above are natural extensions of the definitions of stability of solutions of ordinary differential inequalities given in[5].

3. Let $V(z, y)$ be a complex function analytic in $z \varepsilon D$ and $y \varepsilon C$.

Suppose that

(3·1) $$\left|\frac{\partial V}{\partial y}\right| \leqslant k, \text{ where } k \text{ is a constant.}$$

Define:

(3·2) $$V^*(z, y) = \frac{\partial V}{\partial z} + \frac{\partial V}{\partial y} f(z, y).$$

Assume that

(3·3) $$|V(z, y)| \to o \text{ as } |y| \to o \text{ for each } z \varepsilon D.$$

(3·4) the function $b(r)$ is continuous and non-decreasing in r, $b(r) > o$ for $r > o$ and

$$b(|y|) \leqslant |V(z, y)| \text{ for all } z \varepsilon D.$$

We require the following lemma proved elsewhere[2].

It is stated in a suitable form whose proof needs very little modification of the proof given in[2].

Lemma.—*Let the function $W(t, r)$ be defined and continuous on $I \times R_C^{+}$. Assume that $r(t)$ is the maximal solution of*

(3·5) $r' = W(t, r), \qquad r(t_0) = r_0,$

existing to the right of t_0. Let $m(t)$ be continuous for $t \varepsilon I$ such that $m(t_0) \leqslant r_0$ and

(3·6) $$\limsup_{h \to o^+} \frac{1}{h}[m(t+h) - m(t)] \leqslant W(t, m(t))$$

Then

$$m(t) \leqslant r(t) \qquad (t \geqslant t_0).$$

Now we can prove the following theorems.

Theorem 1.—*Let the function $W_2(t, r)$ be defined and continuous on $I \times R_C^+$. Suppose that the function $V^*(z, y)$ of* (3·2) *satisfies the condition*

(3·7) $$| V^*(z, y) | \leqslant W_2(| z |, | V(z, y) |).$$

Let $r(t)$ be the maximal solution of the differential equation (3·5), *existing to the right of t_0, where*

(3·8) $W(t, r) = kW_1(t, b^{-1}(r)) + W_2(t, r)$, *$k$ being the constant as in* (3·1). *If $y(z)$ be any solution of* (2·1) *such that*

$$| V(z_0, y(z_0)) | \leqslant r_0 \qquad (| z_0 | = t_0),$$

then

$$| V(z, y(z)) | \leqslant r(t) \qquad (z \varepsilon D, | z | = t)$$

for all $t \geqslant t_0$,

For the theorem that follows, we take, $A(z)$ a complex function analytic in $z \varepsilon D$ and define

$$\phi(t) = \min_{\theta} | A(z) | \qquad (z \varepsilon D)$$

(3·10)

$$\psi(t) = \max_{\theta} | A(z) | \qquad (z \varepsilon D)$$

We now state the second theorem.

Theorem 2—*Let the function $W_2(t, r)$ be defined and continuous on $I \times R_C^+$. Suppose that the function $V^*(z, y)$ of* (3·2) *satisfy the condition that*

(3·11) $$| A'(z) V(z, y) + A(z) V^*(z, y) | \leqslant W_2(| z |, | A(z) V(z, y) |).$$

Let $r(t)$ be the maximal solution of the differential equation (3·5) *existing to the right of t_0 where*

(3·12) $$W(t, r) = k \psi(t) W_1\left(t, b^{-1}\left(\frac{r}{\phi(t)}\right)\right) + W_2(t, r)$$

k being the constant as in (3·1). *If $y(z)$ be any solution of* (2·1) *such that*

$$| A(z_0) V(z_0, y(z_0)) | \leqslant r(t_0), \quad (| z_0 | = t_0),$$

then

$$| A(z) V(z, y(z)) | \leqslant r(t) \quad (z \varepsilon D, | z | = t)$$

for all $t \geqslant t_0$.

Proof– Let $L(z, y) = A(z)\, V(z, y)$.

Suppose that $y(z)$ is any solution of (2·1) such that

$$| L(z_0, y(z_0)) | \leqslant r(t_0) \qquad (| z_0 | = t_0, z_0 \in D).$$

From (3·4) and the definition of $\phi(t)$ we note that

$$| y | \leqslant b^{-1}\left(\frac{| L(z, y(z)) |}{\phi(t)} \right). \tag{3·13}$$

For each fixed θ, denote $R(t, .) = | L(z, y(z)) |$.

One can easily verify that

$$\left| \frac{\partial R}{\partial t}(t, .) \right| \leqslant \left| \frac{\partial L}{\partial t}(z, y(z)) \right|$$

But

$$\left| \frac{\partial L}{\partial t}(z, y(z)) \right| = \left| \frac{dL}{dz}(z, y(z))e^{i\theta} \right| = \left| \frac{dL}{dz}(z, y(z)) \right|,$$

therefore

$$\left| \frac{\partial R}{\partial t}(t, .) \right| = \left| \frac{dR}{dt}(t, .) \right| \leqslant \left| \frac{dL}{dz}(z, y(z)) \right| = \left| \frac{\partial L}{\partial z} + \frac{\partial L}{\partial y} y' \right|. \tag{3·14}$$

Moreover, in view of (2·1), (3·1), (3·10), (3·11) and (3·13)

$$\left| \frac{\partial L}{\partial z} + \frac{\partial L}{\partial y} y' \right| \tag{3·15}$$

$$= \left| \left[\frac{\partial L}{\partial z} + \frac{\partial L}{\partial y} y' \right] - \left[\frac{\partial L}{\partial z} + \frac{\partial L}{\partial y} f(z, y(z)) \right] + \left[\frac{\partial L}{\partial z} + \frac{\partial L}{\partial y} f(z, y(z)) \right] \right|$$

$$\leqslant K \,| A(z) |\, W_1(| z |, | y |) + W_2(| z |, | A(z)\, V(z, y(z)) |)$$

$$\leqslant K\, \psi(t)\, W_1\left(| z |, b^{-1}\left(\frac{| L(z, y(z)) |}{\phi(t)} \right) \right) + W_2(| z |, | L(z, y(z)) |).$$

Also for small $h > 0$, we get

$$R(t+h, .) - R(t, .) \leqslant | L(\overline{t+h}\, e^{i\theta}, y(\overline{t+h}\, e^{i\theta})) - L(te^{i\theta}, y(te^{i\theta})) | \tag{3·16}$$

It follows therefore from (3·12), (3·14), (3·15) and (3·16) that

$$\limsup_{h \to 0^+} \frac{R(t+h, .) - R(t, .)}{h} \leqslant W(| z |, R(t, .)).$$

Now application of the Lemma stated above yields the desired result. This proves the theorem 2.

We have given the proof of theorem 2 only since, it can be easily observed that theorem 1 is a particular case of theorem 2. It is stated independently in view of its usefulness in applications.

4. Corresponding to the conditions (*i*), (*ii*) and (*iii*) we shall define the following conditions, where $r(t)$ is any solution of the scalar differential equation (3·5).

(*ia*) For any $\varepsilon > o$ and $t_0 \varepsilon I$, there exists a positive function $d = d(t_0 \; \varepsilon)$ continuous in t_0 for each ε, such that

$$r(t) < \varepsilon \text{ for } t \geqslant t_0$$

whenever $r(t_0) \leqslant d$.

Let $\phi(t)$ be defined as in (3·10).

(*iia*) Given $\varepsilon > o$, $\alpha > o$, $t_0 \varepsilon I$ there is a positive number $T = T(t_0, \varepsilon, \alpha)$ such that

$$r(t) < \varepsilon \phi(t), \qquad (t \geqslant t_0 + T)$$

whenver

$$r(t_0) \leqslant \alpha \phi(t_0).$$

(*iiia*) The conditions (*ia*) and (*iia*) hold simultaneouly.

We now state the following theorem on stability of solutions of complex differential inequality (2·1).

Theorem 3. *Let the assumptions of theorems 1 hold together with the conditions (3·3) and (3·4). If the condition (ia) holds then (i) holds.*

Proof: For any $\varepsilon > o$, if $|y| = \varepsilon$, we deduce from (3·4) that

$$b(\varepsilon) \leqslant |V(z, y)|, \; (z \varepsilon D). \tag{4·1}$$

since the equation (3·8) has the property (*ia*) we have, given $b(\varepsilon) > o$ and $t_0 \varepsilon I$, there exists a positive function $d = d(t_0, \varepsilon)$ that is continuous in t_0 for each ε such that

$$r(t) < b(\varepsilon) \text{ for all } t \geqslant t_0, \tag{4·2}$$

whenever

$$r(t_0) \leqslant d(t_0, \varepsilon). \tag{4·3}$$

Let $y(z)$ be any solution of (2·1).

In view of (3·3) there is a $\delta = \delta(|z_0|, \varepsilon)$ such that

$$\sup_{|y(z_0)| \leqslant \delta} |V(z_0, y(z_0))| \leqslant d. \tag{4·4}$$

Now it follows from theorem 1 that

$$|V(z, y(z))| \leqslant r(t), \; (t \geqslant t_0) \tag{4·5}$$

whenever

$$|V(z_0, y(z_0))| \leqslant r(t_0). \tag{4·6}$$

Let $r(t_0) \leqslant d$. Then it follows from (4·4) and (4·6) that every solution $y(z)$ of (2·1) satisfies (4·5) whenever $|y(z)| \leqslant \delta$.

Let us assume that there exists a solution $y(z)$ of (2·1) for which $|y(z)| = \varepsilon$ for some value of $z = z_1$ such that $|z_1| > |z_0|$. Then using the relations (4·1), (4·2) and (4·5)

we have

$$b(\varepsilon) \leqslant |V(z_1, y(z_1))| \leqslant r(t_1) < b(\varepsilon)$$

a contradiction, which proves the theorem.

Theorem 4 *Let the assumptions of theorem 2 hold together with the conditions* (3·3) *and* (3·4). *If the condition* (*iia*) *holds then* (*ii*) *holds.*

Proof: Let $\varepsilon > o$, $\alpha > o$, and $z_0 \varepsilon D$ be given. Let $y(z)$ be any solution for which $|y(z_0)| \leqslant \alpha$. Then in view of the condition (3·3) there is $\alpha_1 = \alpha_1(|z_0|, \alpha)$ that is continuous in $|z_0|$ for each α, such that

$$(4\cdot7) \qquad \sup_{|y(z_0)| \leqslant \alpha} |V(z_0, y(z_0))| \leqslant \alpha_1.$$

Since the condition (*iia*) holds, given $b(\varepsilon) > o$, $\alpha_1 > o$ and $t_0 \varepsilon I$, there exists a positive number $T = T(t_0, \varepsilon, \alpha)$ such that

$$(4\cdot8) \qquad r(t) < b(\varepsilon)\, \phi(t) \qquad (t \geqslant t_0 + T),$$

whenever

$$(4\cdot9) \qquad r(t_0) \leqslant \alpha_1\, \phi(t_0).$$

Since the conditions of theorem 2 hold, we have

$$(4\cdot10) \qquad |V(z, y(z))|\, \phi(t) \leqslant r(t) \text{ for all } |z| = t \geqslant t_0$$

whenever

$$(4\cdot11) \qquad |V(z_0, y(z_0))|\, \phi(t_0) \leqslant r(t_0).$$

Now choose $r(t_0) \leqslant \alpha_1\, \phi(t_0)$. It follows from (4·7) and (4·11) that whenever $|y(z_0)| \leqslant \alpha$, every solution $y(z)$ of (2·1) satisfies (4·10).

Let $y(z)$ be any solution of (2·1) with $|y(z_0)| \leqslant \alpha$. Choose a sequence $\{z_k\}$ where $|z_k| = t_k \geqslant t_0 + T$ and $t_k \to \infty$ as $k \to \infty$. Suppose that $|y(z_k)| = \varepsilon$, then using the results (4·1), (4·8) and (4·10) we observe

$$b(\varepsilon) \leqslant |V(z_k, y(z_k))| \leqslant \frac{r(t_k)}{\phi(t_k)} < b(\varepsilon)$$

a contradiction. Hence the theorem is proved.

Remark. If the differential equation (3·5) satisfies the condition (*iiia*) then by combining the proofs of theorems 1 and 2 we can prove that the complex differential inequality (2·1) satisfies the condition (*iii*).

We also note that the above results include the stability properties of the solutions of the complex differential equation $y' - f(z, y) = o$.

If, on the other hand, $| g(z, y) | \leqslant W_1(| z | , | y |)$ where $g(z, y)$ is a complex function analytic for each $z \varepsilon D$ and in $y \varepsilon C$, our results show that the solutions of the perturbed differential equation $y' = f(z, y) + g(z, y)$ satisfy the conditions (*i*) to (*iii*).

5. We shall give some examples to illustrate our results.

Example 1. Let D denote the region of the complex plane $| z | > o$ and $o \leqslant \arg z \leqslant 2\pi$. Suppose that $| y | < C$.

Consider the complex differential inequality

(5·1) $$| y' - \frac{y}{2z^2} | \leqslant | y |^2 e^{-| z |}$$

Let $V(z, y(z)) = y^2$. If $b(r) = r^2$, we observe that all the assumption (3·3) and (3·4) are satisfied. From (3·2) and (5·1) we deduce that

$$| V^* | = \left| \frac{V}{z^2} \right|$$

Hence the differential equation (3·8) in view of (3·1) reduces to

(5·2) $$r' = r\left(2c\, e^{-t} + \frac{1}{t^2}\right), \; r(t_0) = r_0,$$

whose solution is given by

$$r(t) = \frac{r_0}{\exp\left[-\frac{1}{t_0} - 2c\, e^{-t_0} \right]} \exp\left[-\frac{1}{t} - 2c\, e^{-t} \right]$$

It is obvious that the condition (*ia*) is satisfied by (5·2). Hence by theorem 3, the inequality (5·1) has the property (*i*).

Example 2. Let D denote the same region as defined in the above example.

Consider the complex differential inequality instead,

(5·3) $$| y' + \frac{y}{2z} | \leqslant | y |^2 e^{-| z |}$$

Taking $V(z, y)$ as before and choosing $A(z) = z$, the equation (3·12) in view of (3·10) reduces to

(5·4) $$r' = 2c\, r\, e^{-t}, \qquad r(t_0) = r_0,$$

the solution of which is given by

$$r(t) = r_0 \exp\left[2c \left\{ e^{-t_0} - e^{-t} \right\} \right]$$

Now observing that $\phi(t) = t$, it is easily seen that the condition (*iia*) is satisfied by (5·4). Hence by theorem 4 (5·3) has the property (*ii*).

Example 3. Let D denote the region of the complex plane $| z | > o$, $-\frac{\pi}{2} \leqslant \arg z \leqslant \frac{\pi}{2}$. Let us suppose that $| y | < C$.

Consider the complex differential inequality

(5·5) $$| y' + y\left(\tfrac{1}{2} + \frac{1}{z} \right) | \leqslant \frac{| y |^2 e^{-| z |}}{| z |^2}$$

Taking $V(z, y)$ and $b(r)$ as before and choosing $A(z) = z^2 e^z$ we note from (3·10) that $\phi(t) = t^2$ and $\psi(t) = t^2 e^t$. From (3·11) and (5·5) it is observed that $W_2(t,r) \equiv o$.

Hence the differential equation (3·12) in view of (3 1) and the definition of $b(r)$ reduces to

$$r' = \frac{2cr}{t^2}, \quad r(t_0) = r_0, \tag{5·6}$$

the solution of which is given by

$$r(t) = r_0 \, e^{2c\left(\frac{1}{t_0} - \frac{1}{t}\right)}$$

Observing that $\phi(t) = t^2$, it is noted that the condition (*iia*) is satisfied by (5·5). Hence by theorem 4, (5·5) has the property (*ii*).

REFERENCES

1. Kayande, A. A. and Lakshmikantham, V. Complex differential systems and extension of Lyapunov's method. *J. Mat Anal. Appl.*, Vol 13 No. 2 : 337-347.
2. Lakshmikantham, V. On the boundedness of solutions of non-linear differential equations. *Proc. A. M S.*, 8 : 1044–48, (1957).
3. Lakshmikantham, V. Differential systems and extension of Lyapunov's Method. *Michigan Jour. Maths*, 9 : 119–126, (1962).
4. Lakshmikantham, V. Properties of solutions of abstract differential inequalities. *Proc. London Math. Soc.*, 14 (53) : 74–82, (1964).
5. Lakshmikantham, V. Differential inequalities and extension of Lyapunov's Method. *Proc. Camb. Phil. Soc*, 60 : (1964).